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Introduction to the Theory of Analytic Spaces

1966

Springer-Verlag · Berlin · Heidelberg · New York
CHAPTER III. LOCAL PROPERTIES OF ANALYTIC SETS

In this chapter, we will be concerned with the local description of analytic sets, both over \( \mathbb{R} \) and over \( \mathbb{C} \). In the first section we shall deal with properties that are valid in either case, and in the second with those properties that are special to complex analytic sets. A more detailed analysis of real analytic sets will be undertaken in Chapter V. The results are mostly contained in Remmert - Stein [32], Cartan [10, 12], Hervé [19].

§ 1. Germs of analytic sets.

Let \( k \) be either \( \mathbb{R} \) or \( \mathbb{C} \), and let \( \Omega \) be an open set in \( k^n \). Analytic functions will mean holomorphic if \( k = \mathbb{C} \), real analytic if \( k = \mathbb{R} \). Let \( S \) be an analytic set in \( \Omega \) and let \( a \in \Omega \). We denote by \( S_a \) the germ of the set \( S \) at \( a \). We refer to \( S_a \) as an analytic germ. Let \( I = I(S_a) \) denote the set of all (germs of) analytic functions in \( \mathcal{O}_{n,a} \) which vanish on the germ \( S_a \) (this statement has an obvious meaning). Clearly \( I \) is an ideal in \( \mathcal{O}_{n,a} \).

We have, obviously, \( S_a \subseteq S_a' \) if and only if \( I(S_a) \supseteq I(S_a') \). We say that \( S_a \) is irreducible if whenever there are two analytic germs \( S_{1a}, S_{2a} \) with \( S_a = S_{1a} \cup S_{2a} \), one of the germs \( S_{1a} \) must be \( = S_a \).

The following lemma is obvious.

Lemma 1. \( S_a \) is irreducible if and only if \( I(S_a) \) is a prime ideal.

Since \( \mathcal{O}_{n,a} \) is noetherian, any increasing sequence of ideals in \( \mathcal{O}_{n,a} \) terminates. Hence any decreasing sequence of analytic germs \( S_{1a}, S_{2a}, \ldots \) terminates. We deduce easily from this the following
Proposition 1. Any analytic germ $S_a$ can be written as a finite union $S_a = \bigcup_{\nu=1}^{k} S_{\nu a}$ of irreducible analytic germs $S_{\nu a}$ such that, for each $\nu$, $S_{\nu a} \cup \bigcup_{\mu \neq \nu} S_{\mu a}$. Further, this decomposition is uniquely determined up to order.

Definition 1. The germs $S_{\nu a}$ introduced by this decomposition $S_a = \cup S_{\nu a}$ are called the irreducible components of $S_a$.

Let now $I$ be an ideal in $\mathcal{O}_n = \mathcal{O}_{n,0}$; we suppose that $\{0\} \neq I \neq \mathcal{O}_n$. If $x_1, \ldots, x_n$ are the coordinates of $k^n$, we shall denote by $\mathcal{O}_p$ the subring of $\mathcal{O}_n$ consisting of functions independent of $x_{p+1}, \ldots, x_n$. We have a natural injection $\mathcal{O}_p \to \mathcal{O}_n$. Let $A$ denote the quotient ring $\mathcal{O}_n/I$. Then, we have a natural homomorphism $\eta : \mathcal{O}_p \to A$.

Proposition 2. After a linear change of coordinates in $k^n$, there is an integer $p$, $0 \leq p < n$, such that $\eta : \mathcal{O}_p \to A$ is injective and makes of $A$ a finite $\mathcal{O}_p$-module.

Proof. Let $f \in I$, $f \neq 0$. We may make a linear transformation of $k^n$ so as to ensure that $f(0,x_n) \neq 0$. This condition is invariant under linear transformations of $k^{n-1}$. By Chapter II, Theorem 2, (2), there is a unit $u$ and a polynomial $q_{n-1}$

$p_n = x_n + \sum_{v=0}^{q_{n-1}} a_v(x_1, \ldots, x_{n-1})x_n^v$, \hspace{1cm} $a_v(0) = 0$, \hspace{1cm} with \hspace{1cm} $f = up_n$;

then $p_n \in I$. Now, either $I_{n-1} = I \cap \mathcal{O}_{n-1} = \{0\}$, in which case we take $p = n - 1$, or there is $f_{n-1} \in I_{n-1} \{0\}$ as above, we find, after a linear change of variables in $k^{n-1}$, that there is a polynomial

$p_{n-1} = x_{n-1} + \sum_{v=0}^{q_{n-1}} a'_v(x_1, \ldots, x_{n-2})x_{n-1}^v$, \hspace{1cm} $a'_v(0) = 0$, \hspace{1cm} $p_{n-1} \in I_{n-1}$.
Continuing this process, we find an integer \( p \) such that \( I_p = I \cap I_p \) = 0 and such that, for any \( r > p \), there is a distinguished polynomial.

\[
p_r = x_r + \sum_{v=0}^{q_{r-1}^{-1}} a_v^{(n-r)}(x_1, \ldots, x_{r-1})x_r^v, \quad a_v^{(n-r)}(0) = 0,
\]

with \( P_r \in I_r = I \cap I_r \).

We claim that this integer \( p \) satisfies our requirements. In fact, trivially \( I_p = \{0\} \) implies that \( \eta : \mathcal{O}_n \to A \) is injective. If \( f \in \mathcal{O}_n \), by Chapter II, Theorem 2, (i), we have

\[
f = \sum_{v=0}^{q_{n-1}^{-1}} f_1, v(x_1, \ldots, x_{n-1})x_n^v (\text{mod } P_n),
\]

\[
f_1, v = \sum_{\mu=0}^{q_{n-1}^{-1}} f_2, \mu(x_1, \ldots, x_{n-2})x_n^{\mu} (\text{mod } P_{n-1}),
\]

and so on, so that

\[
f = \sum_{\alpha_j < q_j} f(\alpha_1, \ldots, \alpha_n)x_1^{\alpha_1 + 1} \ldots x_n^{\alpha_n} (\text{mod } P_{p+1}, \ldots, P_n),
\]

so that the images of the monomials \( x_1^{\alpha_1 + 1} \ldots x_n^{\alpha_n}, \alpha_j < q_j \) generate \( A \) over \( \mathcal{O}_p \).

In what follows, we shall identify elements of \( \mathcal{O}_n \) with their images in \( A = \mathcal{O}_n/I \) when no confusion is likely.

**Corollary.** If, in addition, \( I \) is a prime ideal, \( K \) is the quotient field of \( \mathcal{O}_p \), \( L \) that of \( A = \mathcal{O}_n/I \), then \( L = K(x_{p+1}, \ldots, x_n) \).

**Remark.** The necessary and sufficient condition that the coordinates satisfy the assertion of Proposition 2 is that \( I_p = \{0\} \) and, for \( r > p \), there exists a distinguished polynomial \( Q_r(x_r; x_1, \ldots, x_{r-1}) \in \mathcal{O}_{r-1}[x_r]nI \).
We now state two algebraic theorems that we shall use.

I. (Theorem of primitive element). If $K$ is a field of characteristic zero and $L = K(u_1, \ldots, u_r)$ a finite algebraic extension of $K$, then, for any infinite subset $S \subseteq K$, there exist elements $c_1, \ldots, c_r \in S$ such that

$$L = K(c)$$

where $c = \sum_{i=1}^{r} c_i u_i$.

II. Let $K, L$ be as above, and in addition, suppose that $K$ is the quotient field of a factorial ring $A$, that $B$ is the integral closure of $A$ in $L$, and that $\zeta \in B$ is such that $L = K(\zeta)$. Let $P$ be the minimal polynomial of $\zeta$ over $K$. (Then $P \in A[X]$ since $A$ is factorial.) If $P'$ denotes the derivative of $P$, then for any $\alpha \in B$, there is $Q \in A[X]$ of degree $< $ degree $P$ such that $\alpha P'(\zeta) = Q(\zeta)$ (note that $P'(\zeta) \neq 0$).

Now, by the theorem of primitive element, there exist complex numbers $\lambda_j$ such that $y_{p+1} = \sum_{j=1}^{n} \lambda_j x_j$ is linearly independent of $x_1, \ldots, x_p$ and $L = K(y_{p+1})$. Further, for any $f \in \mathfrak{O}_n$, since $A$ is a finite $A$-module, there exists a polynomial $Q_f(X) = X^m + \sum_{v=0}^{m-1} b_v(x_1, \ldots, x_p)X^v \in A[X]$ with $Q_f(f) = 0$. If we choose the polynomial $Q_f$ to have minimal degree we claim that when $f(0) = 0$, $Q_f$ is a distinguished polynomial. In fact if not all $b_v(0) = 0$, then $X^m + \sum_{v=0}^{m-1} b_v(0)X^v$ has, at $X = 0$, a zero of order $1$. We'll choose $Q(0)$ as the distinguished polynomial, and $Q(X) = X^{l-1} + \sum_{v=0}^{l-1} c_v(x_1, \ldots, x_p)X^v$ is a unit.
a distinguished polynomial of degree 1. But then \( Q(f) = 0 \), and \( Q_f \) would not have minimal degree. Thus we obtain (since \( \mathcal{O}_p \) is factorial)

**Proposition 3.** Given a prime ideal \( I : \mathcal{O}_n', \{0\} \neq I \neq \mathcal{O}_n' \), there exists, after a linear change of coordinates in \( k^n \), an integer \( p, 0 \leq p < n \) such that

\[
\eta : \mathcal{O}_p \to A = \mathcal{O}_n'/I
\]

is an injection which makes \( A \) a finite \( \mathcal{O}_p \)-module. Further, if \( K \) is the quotient field of \( \mathcal{O}_p \), \( L \) that of \( A \), we have \( L = K(x_{p+1}) \), and for any \( r > p \), the minimal polynomial \( P_r \) of \( x_r \) over \( K \) is in \( \mathcal{O}_p[X] \), and is distinguished, so that there is a distinguished polynomial

\[
P_r(x_r; x') = x_r^q + \sum_{v=0}^{q_{r-1}} a_v^{(r)}(x') x_r^v, \quad x' = (x_1', \ldots, x_p'),
\]

\( a_v^{(r)}(0') = 0 \), with \( P_r(x_r, x') \in I \).

It follows that if \( p = 0 \), and if \( I = I(S_0) \), then \( S_0 \) is the germ defined by \( S = \{0\} \).

In what follows, we shall suppose the prime ideal \( I \) given, and the coordinates chosen so that Proposition 3 applies. We shall use the notation of Proposition 3.

Let \( \delta \) denote the discriminant of the polynomial \( P_{p+1} \) (so that \( \delta \) is the resultant of \( P_{p+1} \) and \( \frac{\partial P_{p+1}}{\partial x_{p+1}} \)). Then \( \delta \in \mathcal{O}_p \), further since \( P_{p+1} \) is the minimal polynomial of \( x_{p+1} \) over \( \mathcal{O}_p \), \( \delta \neq 0 \) in \( \mathcal{O}_p \); since \( I_p = \{0\} \), \( \delta \notin I \).

By the algebraic theorem II stated above, if \( q \) is the degree of \( P_{p+1} \), we have the following

**Lemma 2.** For any \( f \in \mathcal{O}_n' \), there is a polynomial \( R_f \) of degree \( < q - 1 \) in \( \mathcal{O}_p[X] \) such that
\[ \delta f - R_f(X_{p+1}) \in I. \]

In particular, there are polynomials \( Q_r \) of degree \( \leq q - 1 \) in \( \mathcal{O}_p[X] \) such that, for \( r > p, \)
\[ \delta x_r = Q_r(x_{p+1}) \in I. \]

**Lemma 3.** For any \( f \in \mathcal{O}_n \), there exists \( g \in \mathcal{O}_n - I \) and \( h \in \mathcal{O}_p \) such that \( gf - h \in I. \)

**Proof.** Since \( A \) is a finite \( \mathcal{O}_p \)-module, we have a relationship
\[ f^m + \sum_{v=0}^{m-1} a_v(x_1', \ldots, x_p') f^v \in I; \]
we may suppose, since \( I \) is prime, that \( a_0(x') \neq 0. \) We have then only to set \( h = -a_0, \) \( g = f^{m-1} + \sum_{v=1}^{m-1} a_v(x') f^{v-1}. \)

**Definition 2.** Let \( S \) be an analytic set in an open set \( \Omega \) in \( k^n. \) A point \( a \in S \) is called a regular point of \( S \) of dimension \( m \) if there is a neighbourhood \( U \) of \( a, U \subset \Omega, \) such that \( S \cap U \) is an analytic submanifold of dimension \( m \) of \( U. \) A point \( a \in S \) is called singular if it is not regular.

A point \( a \in S \) is regular of dimension \( m \) if and only if there exist functions \( f_{p+1}, \ldots, f_n \in \mathcal{O}_{n,a} \) such that, in a neighbourhood of \( a, S = \{ x \mid f_i(x) = 0, i > p \} \) and \( (df_{p+1})_a, \ldots, (df_n)_a \) are linearly independent.

Let \( S \) be an analytic set in an open set \( \Omega \subset k^n, 0 \in S. \) We suppose that \( S_0 \) is irreducible, i.e. that \( I = I(S_0) \) is a prime ideal in \( \mathcal{O}_n = \mathcal{O}_n, 0. \) Choose coordinates in \( k^n \) so that Proposition 3 is satisfied. Then we have

**Proposition 4.** There is a fundamental system of neighbourhoods \( U = U' \times U'' , \quad U' \subset k^p, \quad U'' \subset k^{n-p} \) of \( 0 \) such that if \( x : (S \cap U) \to U' \) denotes the restriction to \( S \cap U \) of the projection of \( U \) onto \( U', \)
then $\pi$ is a proper map and every fibre $\pi^{-1}(x')$, $x' \in U'$, of $\pi$ is a finite set.

**Proof.** Choose a neighbourhood $V$ of 0 such that the polynomials $P_r(x_r, x')$ of Proposition 3 all have coefficients analytic on $V$ and vanish on $S_nV$. Let $V = \{x \mid |x| < \varrho\}$. Since the $P_r$ are distinguished, there is $\sigma > 0$, such that if $|x'| < \sigma$ and $P_r(x_r, x') = 0$, then $|x_r| < \varrho/2$. Clearly if $U' = \{x' \in k^n \mid |x'| < \sigma\}$, $U'' = \{x'' \in k^{n-p} \mid |x''| < \varrho\}$, and $\pi$ is as above, then $
abla^{-1}(E) \subset \{x'' \mid |x''| < \varrho/2\}$, so that $\pi$ is proper. Further, if $x \in \pi^{-1}(x')$, then $P_r(x_r, x') = 0$, and $x_r$ can take at most finitely many different values.

**Lemma 4.** Let $I$ be a prime ideal in $\mathcal{O}_n$, and $S_\varrho$ the germ of analytic set at 0 defined as the set of common zeros of a finite system of generators of $I$. Let $P_r$, $r > p$, $\delta x_r - Q_r(x_{p+1})$ be as in Proposition 3 and Lemma 2. Then there exists a fundamental system of neighbourhoods $U = U' \times U''$ of 0 such that these functions are analytic on $U$, $S_\varrho$ is induced by an analytic set $S$ in $U$, and such that the following hold.

(a) $S_n U \cap \{x \mid \delta(x') \neq 0\} = \{x \in U \mid \delta(x') \neq 0, P_{p+1}(x_{p+1}, x') = 0, \delta x_r - Q_r(x_{p+1}) = 0, r > p + 1\}$.

(b) If $x \in U'$, $x \in k^{n-p}$, and $P_{p+1}(x_{p+1}, x') = 0$, $\delta(x') \neq 0$, then $x \in U$.

**Proof.** Choose $V = V' \times V''$ such that all the functions considered are analytic on $V$; further, let $f_1, \ldots, f_m$ be analytic functions on $V$ with $S_n V = \{x \in V \mid f_i(x) = 0, i = 1, \ldots, m\}$ ($f_i$ generators of $I$). As in the proof of Proposition 2, we find that there exist $f_{\alpha, i} \in \mathcal{O}_p$, $\alpha = (\alpha_{p+1}, \ldots, \alpha_n), \alpha_j < q_j$,
with

$$f_i = \sum_{\alpha_j < q_j} f_{a, i}(x') x_{p+1} \cdots x_n \pmod{p+1, \ldots, p_n}$$

and hence, if $N = q_{p+2} \cdots q_n$ (substitute $\delta x_r = Q_r$)

$$\delta^N f_i = R_i(x_{p+1}) \pmod{p+1, \ldots, p_n, \delta x_{p+2} - Q_{p+2}, \ldots, \delta x_n - Q_n},$$

where $R_i$ is an element in $\mathcal{O}_p[X]$. Again, since $p_{p+1}$ is monic, we may make a polynomial division of $R_i$ by $p_{p+1}$ and obtain

$$\delta^N f_i = R_i(x_{p+1}) \pmod{p_{p+1}, \ldots, p_n, \delta x_{p+2} - Q_{p+2}, \ldots, \delta x_n - Q_n},$$

where $R_i$ is a polynomial of degree $< q - 1$ ($q = \deg p_{p+1}$). Now $f_i \in I$; hence $R_i(x_{p+1}) \in I$. Since $p_{p+1}$ is the minimal polynomial of $x_{p+1}$ over $\mathcal{O}_p$, and $\deg R_i < \deg p_{p+1}$, this implies that $R_i = 0$, so that

$$\delta^N f_i = 0 \pmod{p_{p+1}, \ldots, p_n, \delta x_{p+2} - Q_{p+2}, \ldots}.$$

We now proceed as follows. Clearly, for each $r > p + 1$, we have, on $V' \times k^{n-p}$

$$\delta^q p_r = A'_r(x_{p+1}) \pmod{\delta x_r - Q_r},$$

where $A'_r \in \mathcal{O}_p[X]$. Making a polynomial division of $A'_r$ by $p_{p+1}$, we obtain

$$\delta^q p_r = A_r(x_{p+1}) \pmod{p_{p+1}, \delta x_r - Q_r} \text{ on } V' \times k^{n-p},$$

where $A_r \in \mathcal{O}_p[X]$ and has degree $< \deg p_{p+1}$. Again $A_r(x_{p+1}) \in I$ and so is zero near 0, hence 0 on $V' \times k^{n-p}$. 
Hence

\[(2) \quad \delta^q_{p+1, \delta x - Q} = 0 \pmod{p+1, \delta x - Q} \] on \( V' \times k^{n-p} \). This implies that if \( \delta(x') \neq 0 \), \( P_{p+1}(x_{p+1}, x') = \delta x - Q = 0 \), then \( P_{p+1}(x_{p+1}, x') = 0 \). Since \( P_{p+1} \) is distinguished, this implies that if \( V' \) is small, then any solution \( x \in V' \times k^{n-p} \) of \( P_{p+1}(x_{p+1}, x') = 0 = \delta x - Q \), \( r > p + 1 \), lies in a preassigned neighbourhood \( V' \times V'' \) of 0. This proves (b).

Again, (1) and (2) imply that, for a fixed integer \( M > 0 \),

\[(3) \quad \delta^M f = 0 \pmod{p+1, \delta x_{p+2} - Q_{p+2}, \ldots, \delta x_n - Q_n} \].

If we now choose \( U \subset V \), \( U = U' \times U'' \) such that (b) holds, and all the above congruences are represented by linear relations with coefficients analytic on \( U \), then

\[ S \cap U \cap \{ x \mid \delta(x') \neq 0 \} = \{ x \in U \mid \delta(x') \neq 0, f_1(x) = \ldots = f_m(x) = 0 \}, \]

and, by (3), this is

\[ = \{ x \in U \mid \delta(x') \neq 0, P_{p+1}(x_{p+1}, x') = 0 = \delta x - Q_{p+1}(x_{p+1}), r > p + 1 \}. \]

This proves Lemma 4.

We remark that in Proposition 4 and Lemma 4, given any \( V' \subset k^{n-p} \) (which is a neighbourhood of 0), then we can find a neighbourhood \( V' \) of 0 in \( k^p \) such that for any open set \( U' \subset V', 0 \in U' \), the assertions in Proposition 4 and Lemma 4 are true.

**Proposition 5.** Let \( U \) be a neighbourhood of 0 such that Lemma 4 is true relative to the ideal \( I = I(S_0) \) where \( S_0 \) is irreducible. Then any point \( x \in S \cap U \) with \( \delta(x') \neq 0 \) is a regular point of \( S \) of dimension \( p \), and the projection \( \pi \) has a jacobian of rank \( p \) at \( x \).
Proof. Since $\delta(x') \neq 0$ and $P_{p+1}(x) = 0$, we conclude that, at the point $x$, $\frac{\partial P_{p+1}}{\partial x} \neq 0$. Hence, $S$ is defined near $x$ by the system of equations

$$P_{p+1}(x_{p+1}, x') = 0, x_r = \frac{Q_r(x_{p+1})}{\delta(x')}, r > p + 1,$$

which has the property that $d_{p+1} ... d\left(x_r - \frac{Q_r}{\delta(x')}\right)$ are $k$-independent at $x$. This proves Proposition 5.

Remark further that if $X, Y$ are analytic manifolds countable at $\infty$ and $f : X \to Y$ is an analytic map such that $f^{-1}(y)$ is discrete for every $y \in f(X)$, then $\dim X + \dim Y$ (apply the rank theorem to a point where the differential of $f$ has maximal rank). Combining this with Proposition 5 we obtain

**Proposition 6.** The integer $p$ of Proposition 2 and 3 relative to $I = I(S_a)$, $S_a$ being irreducible, is the largest integer $m$ such that $(S_a)$ is induced by an analytic set $S$), every neighbourhood of $0$ contains points at which $S$ is regular of dimension $m$.

This characterisation of the integer $p$ is clearly invariant under analytic automorphism of a neighbourhood of $0$ in $k^n$.

**Definition 3.** The dimension of an irreducible analytic germ $S_a$ at $a \in k^n$ is the integer $p$ of Proposition 2. The dimension of an arbitrary analytic germ $S_a$ is the maximum dimension of of the irreducible components $S_{v,a}$ of $S_a$. The dimension of an analytic set $S$ in an open set $\Omega$ in $k^n$ is $\max_{a \in S_a} \dim_{S_a}$, where $S_a$ is the germ at a defined by $S$. 
Theorem 1. Let $S$ be an analytic set in an open set $\Omega$ in $k^n$. Let $a \in S$ and $\dim S_a = p$. Then any neighbourhood of $a$ contains points at which $S$ is regular of dimension $p$. In particular, the set of regular points of $S$ is dense in $S$.

Proof. Let $T_a$ be an irreducible component of $S_a$ of dimension $p$. Let $T'_a$ be the union of the other irreducible components of $S_a$. Let $T''_a = T'_a \cap T'_a$ and $T, T''$ be analytic sets in an open set $U$ containing $a$ inducing the germs $T_a, T''_a$ at $a$; then $S \cap U = T \cap T''$. Since a regular point of $T$ of dimension $p$ which does not lie on $T''$ is clearly a regular point of $S$ of dimension $p$, it suffices to prove that $U$ contains a regular point of $T$ of dimension $p$ not on $T''$. Let $U$ be so chosen that Propositions 3, 4, 5 apply to $T_a$. Then, there is $f \in \mathcal{O}_n$, $f = 0$ on $T''_a$, $f \notin I = I(T_a)$ (since $T''_a \cap T'_a$). Let $\delta$ have the significance of Proposition 4 (relative to $T_a$). Then $\delta \notin I$. Since $I$ is prime, $f \notin \delta I$.

Hence, arbitrarily near $a$, there are points $x \in T$ with $f(x) \neq 0$. Theorem 1 follows from Proposition 5.

Proposition 7. If $S_a$ is an irreducible germ of dimension $p$ and $S_a \supset S'_a$, where $S'_a$ is any analytic germ at $a$, then $\dim S'_a < \dim S_a$.

Proof. Clearly, we may suppose that $S'_a$ is irreducible. We choose the coordinates $x_1, \ldots, x_n$ in $k^n$ so that if $I = I(S'_a)$, $I' = I(S'_a)$ (so that $I \subset I'$), then we have $I_p = \{0\}$. There exists a distinguished pseudopolynomial $P_r(x_r; x') \in I$, $r > p$. If we show that, after a linear change of variables in $k^p(x_1, \ldots, x_p)$, there is a distinguished polynomial $P_p(x_p; x_1, \ldots, x_{p-1}) \in I'$, the result follows from the remark after Proposition 2. Now, by the preparation theorem, it suffices to prove that there exists $h \in \mathcal{O}_n \setminus I' = I''$, $h \neq 0$. Let $g \in \mathcal{O}_n$, $g \notin I$, $g \in I'$. By Lemma 3, there is $g_1 \notin \mathcal{O}_n - I$
such that $g_g \equiv h \mod I$, where $h \in \mathcal{O}_p$. But then clearly, since $g \in I'$, $h \in I'$, and, since $I$ is prime, $g_g \not\equiv I$, so that $h \not\equiv I$, and in particular $h \neq 0$.

§ 2. Complex analytic sets.

In this section we shall deal only with complex analytic sets, so that $k = \mathbb{C}$.

Let $I$ be a prime ideal in $\mathcal{O}_{n,a} = \mathcal{O}_n$, and suppose that the coordinates $(x_1, \ldots, x_n)$ are so chosen that Proposition 3, 4 and Lemma 4 are valid. Let $\pi : S_n U \to U'$ be the projection defined in Proposition 4.

For any ideal $I \subseteq \mathcal{O}_{n,a}$ we denote by $S(I)$ the germ at $a$ of analytic set defined as the set of common zeros of a finite system of generators of $I$. Clearly, $S(I)$ is independent of the system of generators chosen.

**Proposition 8.** We have $\pi(S_n U) = U'$.

**Proof.** Since $\pi$ is proper, its image is closed in $U'$.

Hence it suffices to show that $\pi(S_n U)$ is dense in $U'$.

For any $x' \in U'$, $\delta(x') \neq 0$, the polynomial $p_{p+1} (x_{p+1}, x')$ has a complex zero $x_{p+1}$. Let $x = \left( x', x_{p+1}, \frac{q_{p+2}(x_{p+1})}{\delta(x')}, \ldots, \frac{q_n(x_{p+1})}{\delta(x')} \right)$.

By Lemma 4, $x \in S_n U$, and clearly $\pi(x) = x'$. Hence $\pi(S_n U)$ contains the dense set $\{ x' \in U' | \delta(x') \neq 0 \}$.

**Remarks.** 1. Note that, by the remark following Proposition 3, this implies that, if the coordinates are so chosen that Proposition 2 is valid, then there is a fundamental system of neighbourhoods $\{ U_v \}$ of $0$ such that $\pi_p(S_n U_v)$ is a neighbourhood of $0$ in $\mathbb{C}^p$, $\pi_p$ being the projection of $\mathbb{C}^n$ onto $\mathbb{C}^p$.

2. Actually, the map $\pi : S_n \{ x \in U | \delta(x') \neq 0 \} \to U' - \{ x' \in U' | \delta(x') \neq 0 \}$ is a covering map.
Lemma 5. Let \( f \in \mathcal{O}_n \). If, for sufficiently small \( U \) as above, for any \( x' \in U' \), \( \delta(x') \neq 0 \), there is \( x \in \mathcal{S}_n U \) such that \( \pi(x) = x' \) and \( f(x) = 0 \), then \( f \in \mathcal{I} \).

Proof. If \( f \notin \mathcal{I} \), there is \( g \in \mathcal{I} \) such that \( gf = h( \text{mod} \ I) \) where \( h \in \mathcal{O} \). Then \( h \in \mathcal{I} \), and for any sufficiently small \( x' \), \( \delta(x') \neq 0 \), \( h(x') = f(x) g(x) = 0 \) (if \( x \in \mathcal{S}_n U \), \( \pi(x) = x' \)), so that \( h = 0 \) and so \( h \notin \mathcal{I} \), a contradiction.

Theorem 2. (Hilbert's Nullstellensatz). Let \( \mathfrak{a} \) be any ideal of \( \mathcal{O}_n \) and \( \mathcal{S}_O = S(\mathfrak{a}) \) the germ of analytic set defined as the set of common zeros of a finite system of generators of \( \mathfrak{a} \). Then \( I(\mathcal{S}_O) = \mathrm{rad}\mathfrak{a} = \{ f \in \mathcal{O}_n | f^m \in \mathfrak{a} \text{ for some integer } m > 0 \} \).

Proof. We first remark that, if \( \mathfrak{a} \) is prime, \( I(\mathcal{S}_O) = \mathfrak{a} \). This is a trivial consequence of Lemma 5. Hence, if \( \mathfrak{a} \) is primary (i.e. \( \mathrm{rad}\mathfrak{a} \) is prime), we deduce, since \( S(\mathfrak{a}) = S(\mathrm{rad}\mathfrak{a}) \), that \( I(\mathcal{S}_O) = \mathrm{rad}\mathfrak{a} \). If \( \mathfrak{a} \) is arbitrary, \( \neq \{0\} \), since \( \mathcal{O}_n \) is noetherian, we obtain by the Noether decomposition theorem,

\[
\mathfrak{a} = \bigwedge_{v=1}^{k} \mathfrak{q}_v, \quad \mathfrak{q}_v \text{ being primary.}
\]

Clearly then

\[
\mathcal{S}(\mathfrak{a}) = \bigcup_{v=1}^{k} \mathcal{S}(\mathfrak{q}_v),
\]

so that

\[
I(\mathcal{S}(\mathfrak{a})) = \bigcap_{v=1}^{k} I(\mathcal{S}(\mathfrak{q}_v)) = \bigcap_{v=1}^{k} \mathrm{rad}\mathfrak{q}_v = \mathrm{rad}\mathfrak{a}.
\]

We now give a very important application of the results obtained above. We begin with a definition.

Definition 4. Let \( S \) be an analytic set in an open set \( \Omega \) in \( \mathbb{C}^n \). A function \( f \) on \( S \) is said to be holomorphic at \( a \in S \) if there is a neighbourhood \( U \) of \( a \) in \( \Omega \) and a holomorphic function \( F \) in \( U \) with \( F|_{U \cap S} = f|_{U \cap S} \).
We may define germs of holomorphic functions in the obvious way. If \( a \in S \), let \( \mathcal{O}_{S, a} \) denote the ring of germs of holomorphic functions at \( a \) on \( S \). Clearly, we have

\[
\mathcal{O}_{n,a}/I(S_a) \cong \mathcal{O}_{S, a}.
\]

Hence \( \mathcal{O}_{S, a} \) is an analytic ring over \( \mathbb{C} \).

**Definition 5.** A map \( f : S_1 \to S_2 \) (\( S_1 \) analytic set in an open set in \( \mathbb{C}^{n_1} \)) is called holomorphic if the map \( j \circ f \), where \( j : S_2 \to \mathbb{C}^{n_2} \) is the natural injection, has the form \( j \circ f = (f_1, \ldots, f_{n_2}) \) where the \( f_v \) are holomorphic on \( S_1 \).

Clearly, a holomorphic map \( f : S_1 \to S_2 \) induces, for \( a \in S_1 \), an algebra homomorphism

\[
f^* : \mathcal{O}_{S_2, f(a)} \to \mathcal{O}_{S_1, a}
\]

viz, \( f^*(\varphi) = \varphi \circ f \).

**Theorem 3.** Let \( f : S_1 \to S_2 \) be holomorphic. Then the homomorphism

\[
f^* : \mathcal{O}_{S_2, f(a)} \to \mathcal{O}_{S_1, a}
\]

is finite (see Chapter II) if and only if \( a \) is an isolated point of the fibre \( f^{-1} f(a) \).

**Proof.** Let \( S_1 \cap \mathbb{C}^n(x_1, \ldots, x_n), \ S_2 \cap \mathbb{C}^m(y_1, \ldots, y_m) \).

We may suppose that \( a = 0, \ f(a) = 0 \). We set

\[
R_1 = \mathcal{O}_{S_1, 0}, \quad R_2 = \mathcal{O}_{S_2, f(0)}.
\]

Suppose that

\[
f^* : R_2 \to R_1
\]

is finite. Then every element of \( R_1 \) is integral over \( R_2 \).
Hence, if $\varphi \in \mathcal{O}_{S_1,0}$, there exist holomorphic germs $a_1, \ldots, a_r \in \mathcal{O}_{S_2,0}$ such that

$$\varphi^r(x) + \sum_{v=1}^{r} a_v(f(x))\varphi^{r-v}(x) = 0.$$ 

In particular, we have, in some neighbourhood of 0 on $S_1$,

$$x_k^r + \sum_{v=1}^{r} a_v^{(k)}(f(x))x_k^{r-v} = 0, \quad k = 1, \ldots, n, a_v^{(k)} \in \mathcal{O}_{S_2,0}.$$ 

Hence, if $f(x) = 0$, and $x$ is near 0 on $S_1$, $x_k$ satisfies a polynomial relation and so can be at most one of finitely many complex numbers. Hence 0 is isolated in $f^{-1}f(0)$.

Suppose conversely that 0 is an isolated point of $f^{-1}f(0)$. This means precisely that if $\mathfrak{a}$ is the ideal of $\mathcal{O}_{n,0}$ generated by $(f_1, \ldots, f_m, I(S_0))$, then $S(\mathfrak{a}) = \{0\}$. (For notation $S(\mathfrak{a})$, see Theorem 2). Hence, by the Nullstellensatz, for any $k, 1 \leq k \leq n$, there is an integer $r$ such that

$$x_k^r = \sum_{v=1}^{m} a_v(x)f_v(x) \pmod {I(S_0)}, \quad a_v \in \mathcal{O}_{n,0}.$$ 

This implies clearly that there is an integer $q > 0$ such that $[\mathbb{R}_1]^{q} f^*([\mathbb{R}_2]) \cdot R_1$ (for notation see Chapter II).

This implies that $f^*: R_2 \rightarrow R_1$ is quasi-finite. By Theorem 1, Chapter II, $f^*$ is finite, q.e.d.

**Corollary 1.** The necessary and sufficient condition that a system of coordinates $(x_1, \ldots, x_n)$ of $\mathbb{C}^n$ satisfy the assertion of Proposition 2 relative to an ideal $I \mathcal{O}_n$ is that 0 is an isolated point of the set

$\{x_1 = \ldots = x_p = 0\} \cap S(I)$ and $I \mathcal{O}_p = \{0\}$. 

Corollary 2. If $X, Y$ are analytic sets in open sets in $\mathbb{C}^n$, $\mathbb{C}^m$ respectively, $f : X \to Y$ a holomorphic map for which $a \in X$ is an isolated point of $f^{-1}f(a)$, then there is a neighbourhood $U$ of $a$ such that any $b \in U$ is an isolated point of $f^{-1}f(b)$.

Proof. Let $X$ be an analytic set in an open set $\Omega$ in $\mathbb{C}^n$ and suppose that $a = 0$. By Theorem 3, there exist $\alpha_{\mu, i} \in \mathbb{O}, b, b \in f(0)$, such that

$$x_i^{p_i} - \sum_{\mu=1}^P \alpha_{\mu, i} f(x) x_i^{p_i-\mu} = 0 \text{ in } \Omega_{X, 0},$$

here $x_1, \ldots, x_n$ are the coordinates in $\mathbb{C}^n$. There is a neighbourhood $U$ of 0 and an open set $V \subseteq Y, b \in V, b \subseteq f(U)$, such that $a_{\mu, i}$ are holomorphic in $V, f(U) \cap V$, and the above equations hold on $U$. Then, given $f(x) \in V$, each $x_i$ can have only finitely many values, which proves our assertion.

Corollary 3. If $S$ is an analytic set in an open set $\Omega \subseteq \mathbb{C}^n$ of dimension $p$ at every point and the restriction to $S$ of the projection $\pi$ of $\mathbb{C}^n$ onto $\mathbb{C}^p$ is a proper map with finite fibres into $\Omega' = \pi(\Omega)$, then $\pi|S$ is an open map.

Proof. Let $0 \in S$ and $U$ a neighbourhood of 0; we have to show that $\pi(U \cap S)$ is a neighbourhood of $0 = \pi(0) \in \mathbb{C}^p$. This is an immediate consequence of Corollary 1 above and the Remark 1 after Proposition 8.

Corollary 4. If $S$ is an analytic set in an open set $\Omega$ in $\mathbb{C}^n$ and $0 \in S$, then $\dim S_0$ is the smallest integer $k$ such that there exists a subspace $H$ of $\mathbb{C}^n$ of dimension $n - k$ such that 0 is an isolated point of $H \cap S$.

This follows easily from Corollary 1 above and the definition of $\dim S_0$.

Corollary 5. If $X$ is an analytic set in an open set in $\mathbb{C}^n$ and $f : X \to \mathbb{C}^k$ is a holomorphic map, then any point $a \in X$
has a neighbourhood $U$ in $X$ such that, for $x \in U$

$$\dim_X f^{-1}f(x) \leq \dim_a f^{-1}f(a).$$

**Proof.** If $p = \dim_a f^{-1}f(a)$, then, if the coordinates at $a$ are suitably chosen, $a$ is an isolated point of the set

$$f^{-1}f(a) \cap \left\{ z \in \mathbb{C}^n \mid z_1 = \ldots = z_p = 0 \right\}.$$

Let $g : X \to \mathbb{C}^{k+p}$ be the map $g(z) = \left(f(z), z_1, \ldots, z_p\right)$. Then $a$ is an isolated point of $g^{-1}g(a)$; by Corollary 2 above, there is an open set $U'$ in $\mathbb{C}^n$ containing $a$ such that, for $z \in U \cap X$, $x$ is an isolated point of $g^{-1}g(x)$. This means precisely that $x$ is an isolated point of

$$f^{-1}f(x) \cap \left\{ z \in \mathbb{C}^n \mid z_1 = \ldots = z_p = 0 \right\}.$$

By Corollary 4 above, $\dim_X f^{-1}f(x) \leq p$.

We now continue our study of complex analytic sets.

**Theorem 4.** Let $\Omega$ be an open set $\mathbb{C}^n$ and $\pi$, the restriction to $\Omega$ of the projection of $\mathbb{C}^n$ onto $\mathbb{C}^p$ (first $p$ variables). Let $\Omega' = \pi(\Omega)$ and $A'$ be a thin subset of $\Omega'$. Let $X$ be an analytic set in $\Omega - \pi^{-1}(A')$ and suppose that $\pi|X$ is a finite covering of $\Omega' - A'$ (i.e. $\pi|X$ is proper and locally biholomorphic) and that $\pi|X$ is a proper map into $\Omega'$. Then $\overline{X}$ is an analytic set in $\Omega$ of dimension $p$ at each of its points.

**Proof.** Let $A = \pi^{-1}(A')$. We may suppose that $\Omega$ (and hence $\Omega'$) is connected. Hence $\Omega' - A'$ is connected (Chapter I, Proposition 11) and hence there is an integer $k$ such that for any $x' \in \Omega' - A'$, there are exactly $k$ points $x^{(1)}, \ldots, x^{(k)} \in X$ with $\pi(x^{(j)}) = x'$.

Let $f$ be any holomorphic function on $\Omega$. We define holomorphic functions $a_1, f, \ldots, a_k, f$ on $\Omega'$ as follows.
For \( x' \in \Omega' - A' \), let \( a_{l,f} \) be the \( l \)-th elementary symmetric function

\[
a_{l,f}(x') = (-1)^l \sum_{1 \leq j_1 < \ldots < j_l \leq k} f(x^{(j_1)}) \ldots f(x^{(j_l)}),
\]

where \( x^{(1)}, \ldots, x^{(k)} \) are the points of \( X \) with \( \pi(x^{(j)}) = x' \).

Clearly \( a_{1,f} \) is holomorphic in \( \Omega' - A' \), and further, since \( \pi : \overline{X} \to \Omega' \) is proper, for any compact set \( K' \subset \Omega' \), \( a_{1,f} \) is bounded on \( K' - A' \), and hence (Chapter I, Proposition 10) can be extended to a holomorphic function on \( \Omega' \). Let \( P_f \) be the holomorphic function on \( \Omega \) defined by

\[
P_f(x) = f^k(x) + \sum_{l=1}^k a_{l,f}(x') f^{k-l}(x), \quad \pi(x) = x'.
\]

By construction, \( P_f(x) = 0 \) if \( x \in X \), and hence \( P_f(x) = 0 \) if \( x \in \overline{X} \). We claim that

(4) \( \overline{X} = \{ x' \in \overline{\Omega} | P_f(x') = 0 \text{ for any holomorphic } f \text{ on } \Omega \} \).

Let \( X' \) be the set of common zeros of the \( P_{f'} \), let \( x' \in \Omega' \), and set \( E' = \{ x' \in X' | \pi(x) = x' \} \), \( \overline{E} = \{ x \in \overline{X} | \pi(x) = x' \} \). Then \( \overline{E} \subset E' \), and to prove (4), it suffices to prove that \( \overline{E} = E' \).

For this, it suffices to prove that \( f(\overline{E}) = f(E') \) for any holomorphic \( f \) on \( \Omega \). Let \( \alpha \in f(E') \); because of the continuity of the roots of a polynomial, there is a sequence \( x'_v \) of points, \( x'_v \to x' \), \( x'_v \in A' \), such that there is a zero \( \alpha_v \) of the polynomial \( \zeta^k + \sum_{l=1}^k a_{l,f}(x'_v) \zeta^{k-l} \) such that the sequence \( \alpha_v \to \alpha \); but since \( x'_v \in A' \), there is \( x_v \in X \), \( \pi(x_v) = x'_v \), with \( \alpha_v = f(x_v) \). Further, since \( \pi : \overline{X} \to \Omega' \) is proper, we can find a subsequence \( \{ x_{v_k} \} \) such that \( x_{v_k} \to x \in \overline{X} \); clearly then \( x \notin E \) and \( f(x) = \lim f(x_{v_k}) = \alpha \).

Hence \( \alpha \in f(\overline{E}) \), and (4) is proved. Because of Corollary 2 to Theorem 5, Chapter II, \( \overline{X} \) is analytic in \( \Omega \). It is clear, since \( \pi : X \to \Omega' - A' \) is locally biholomorphic, that
$X$ is regular of dimension $p$ at every point of $X$. Since $X$ is dense in $X$, Proposition 5 implies that $X$ has dimension $p$ at every point.

**Proposition 9.** Let $S$ be an analytic set in $\Omega$ such that $S_0$ is irreducible. Let $U$ be a neighbourhood of $0$ as in Proposition 4 and Lemma 4. Then the set $X = \{x \in S|\delta(x') \neq 0\}$ is dense in some neighbourhood of $0$ on $S$.

**Proof.** We have already remarked that the projection $\pi: X \to U' = \{x' \in U'|\delta(x') \neq 0\}$ is a covering. Further $\pi: S \to U'$ is proper, hence so is $\pi: X \to U'$. Hence, by Theorem 4, $X$ is an analytic set of dimension $p$ at $0$. But since $X \subseteq S$, Proposition 7 implies that there is a neighbourhood $V$ of $0$ with $X \cap V = S \cap V$.

It is also possible to avoid Proposition 7 and use directly the argument used in Theorem 4. This method leads, in fact, to a somewhat stronger form of Proposition 9. Of course, Proposition 9 is stronger than Propositions 5 and 6.

If $f \in O_{\Omega,0}$, $f \in I(S_0)$, then there is $g \in I(S_0)$, $gf \equiv h \pmod{I(S_0)}$ where $h \in O_p$. Replacing $X$ in the above proof by the set $\{x \in S|h(x')\delta(x') \neq 0\}$, we deduce

**Proposition 9'.** If $S$ is as in Proposition 9 and $f \in I(S_0)$, then the set of points $\{x \in S|f(x) = 0\}$ is dense in some neighbourhood of $0$ in $S$.

It is trivial matter to extend Proposition 9' to sets $S$ for which $S_0$ is not necessarily irreducible (with the obvious conditions on $f$).

**Definition 6.** Let $S$ be an analytic set in an open set $\Omega$ in $C^n$. A holomorphic function $f$ on $\Omega$ is called a universal denominator for $S$ at a point $a \in S$ if $a$ has
a neighbourhood $U$ in $\Omega$ such that the following holds: if $h$ is a holomorphic function on the set $S'$ of points of $S \cap U$ at which $S$ is regular, and if $h$ is bounded on $S'$, then there is a neighbourhood $V$ of $a$ such that $fh$ is the restriction to $S' \cap V$ of a holomorphic function on $V$.

**Theorem 5.** Let $S$ be an analytic set in $\Omega$, and suppose that $S$ has dimension $p$ at each point. Let $\pi$ denote the projection of $\mathbb{C}^n$ onto $\mathbb{C}^p$ (first $p$ variables), and let $\Omega' = \pi(\Omega)$. Suppose that $\pi|S$ is a proper mapping into $\Omega'$ with finite fibres (i.e. $\pi^{-1}(x') \cap S$ is finite for any $x' \in \Omega'$). Then, given a point $a \in S$ which is regular on $S$ at which $x_1 - a_1, \ldots, x_p - a_p$ form a system of local coordinates, there exists a linear function $l$ on $\mathbb{C}^n$ and holomorphic functions $\alpha_1, \ldots, \alpha_k$ on $\Omega'$ such that, if we set

$$P(t, x') = t^k + \sum_{\nu=1}^{k} \alpha_{\nu}(x') t^{k-\nu},$$

the following conditions are satisfied.

(a) $P(l(x), \pi(x)) \equiv 0$ on $S$.

(b) $\frac{\partial P}{\partial t}(l(a), \pi(a)) \neq 0$, $P'(x) = \frac{\partial P}{\partial t}(l(x), \pi(x))$ is a universal denominator for $S$ at every one of its points.

(c) If $h$ is holomorphic and bounded on the set $S'$ of regular points of $S$ there exist holomorphic functions $\beta_0, \ldots, \beta_{k-1}$ on $\Omega'$ such that

$$P'(x)h(x) = \sum_{\nu=0}^{k-1} \beta_{\nu}(\pi(x))(l(x))^\nu$$

on $S'$, and there is a constant $M$ (independent of $h$) such that

(e) $\|\beta_{\nu}\|_{\Omega'} \leq M \|h\|_{S'}$. 

Proof. We begin by proving the following. There exists a thin subset \( A' \) of \( \Omega' \) such that if \( A = S \cap \pi^{-1}(A') \) then \( \pi : S - A \rightarrow \Omega' - A' \) is a (finite) covering. Note that, by Corollary 3 to Theorem 3, \( A \) is nowhere dense in \( S \).

Let \( B \) be the union of the set of singular points of \( S \) with the set of points of \( S' \) where the jacobian matrix of the map \( \pi : S' \rightarrow \Omega' \) is not invertible. (Since \( \pi : S' \rightarrow \Omega' \) has finite fibres, this is a nowhere dense analytic set in \( S' \).)

Clearly, \( B \) is closed, hence so is \( \pi(B) = A' \). We claim that \( A' \) is thin. For this, it is clearly sufficient to prove the following: if \( b' \in \Omega' \) and \( b \in S \), \( \pi(b) = b' \), then, there is a neighbourhood \( U \) of \( b \) and a thin set \( E' \) in a neighbourhood \( U' \) of \( b' \) in \( \Omega' \) such that \( S \) is regular at any point of \( \bigcup_{b \in U} \pi^{-1}(E') \), and \( \pi|_{\bigcup_{b \in U} \pi^{-1}(E')} \) is of maximal rank. Let \( S_{v,b} \) be the irreducible components of \( S \) at \( b \). By Corollary 1 to Theorem 3 and Lemma 4, there is a holomorphic function \( g'_{v,b} \) near \( b' \) (which is a multiple of the discriminant \( \delta_v \) corresponding to \( \pi_{b} \)) such that \( g'_{v,b} \) is a local homeomorphism, it is a finite covering.

We now choose a linear function \( l \) on \( \mathbb{C}^n \) such that:

(i) for a (countable) dense set of points \( x' \in \Omega' - A' \), \( l \) separates the points of \( \pi^{-1}(x') \); 
(ii) for the given point \( a \), \( l(a) \) is different from \( l(c) \) for any \( c \neq a \), \( c \in \pi^{-1}(a) \).

Suppose that \( \pi : S - A \rightarrow \Omega' - A' \) is a covering of \( k \) sheets (note that, by the Corollary to Theorem 3 and Proposition 8
\(\pi : S \to \Omega'\) is an open map. Let \(\alpha_1, \ldots, \alpha_k\) be the holomorphic functions on \(\Omega' - A'\) which are the elementary symmetric functions of the values of \(l\) on the fibres of \(\pi\). They have holomorphic extensions to \(\Omega'\). We set then

\[
P(\xi, x') = \xi^k + \sum_{v=1}^{k} \alpha_v(x') \xi^{k-v}.
\]

Clearly, \(P(l(x), l(x)) = 0\) on \(S - A\), hence, this latter set being dense in \(S_j = 0\) on \(S\). This is (a).

Since by assumption \(x_1 - a_1, \ldots, x_p - a_p\) form local coordinates at \(a\) on \(S\), \(\pi\) is a homeomorphism in a neighbourhood of \(a\); hence, by our assumption (ii), \(l(a)\) is a simple root of \(P(\xi, a')\), and hence

\[
\frac{\partial P}{\partial \xi}(l(a), l(a)) \neq 0.
\]

To complete the proof of Theorem 5, it suffices now to prove (c); to obtain the second part of (b), we have only to apply (c) to a neighbourhood of a given point of \(S\).

We have only to find holomorphic functions \(\beta_v\) on \(\Omega'\) such that (e) holds and (d) holds on \(S - A\). Let \(x' \in \Omega' - A'\), and let \(x^{(1)}, \ldots, x^{(k)}\) be the points of \(S - A\) with \(\pi(x^{(j)}) = x'\). Consider the sum

\[
\sum_{j=1}^{k} \frac{P(\xi; x^{(j)}) - P(l(x^{(j)}); x')}{\xi - l(x^{(j)})} h(x^{(j)})
\]

this is clearly of the form

\[
\sum_{v=0}^{k-1} \beta_v(x') \xi^v
\]

where the \(\beta_v\) are holomorphic on \(\Omega' - A'\), and for any \(x'\), \(\beta_v(x')\) is a linear combination of the \(h(x^{(j)})\) with coefficients depending only on the \(l(x^{(j)})\). Hence

\[
|\beta_v(x')| < M \max_{j} |h(x^{(j)})|.
\]
In particular, the \( \beta_v \) are bounded on \( \Omega' - A' \) (since \( h \) is bounded on \( S' \)) and so admits a holomorphic extension
to \( \Omega' \) such that (e) holds. If in the identity
\[
\sum_{j=1}^{k} \frac{P(t;x')}{\xi - 1(x^{(j)})} h(x^{(j)}) = \sum_{v=0}^{k-1} \beta_v (x') \xi^v
\]
we substitute \( \xi = l(x) \) where \( x \) is a point such that \( \pi(x) = x' \in \Omega' - A' \) and \( l \) separates the points of \( \pi^{-1}(x') \)
[such points are dense in \( S - A \)], we obtain (d) on a
dense subset of \( S - A \); hence (d) holds on \( S' \).

**Remark.** The set of points where \( S \) is not regular is
clearly contained in the set \( P'(x) = \frac{\partial P}{\partial \xi} (l(x),\pi(x)) = 0. \)

**Corollary.** If \( S \) is an analytic set such that \( S_0 \) is
irreducible there exists \( f \in \mathcal{O}_{n_0} - I(S_0) \) which is a
universal denominator at any point sufficiently near 0.

Of course, this corollary can be proved directly in a
somewhat simpler fashion. In fact, the last part of our
arguments shows that with the notation of Proposition 3,
\( \frac{\partial P}{\partial x^{p+1}} \)
is such a universal denominator.

We shall see later that the finiteness of the fibres
of \( \pi | S \) is a consequence of the hypothesis that \( \pi | S \) is
proper.

**Remark.** Theorem 4 remains valid if we replace the assumption
that \( \pi : X \rightarrow \Omega' - A' \) is an unramified covering by the
assumption that \( \pi : X \rightarrow \Omega' - A' \) is a proper map with finite
fibres and that \( X \) has dimension \( p \) at each point. In fact,
we have seen in the proof of Theorem 5 that there is a thin
subset \( B' \subset \Omega' - A' \) such that \( X - \pi^{-1}(B') \) is dense in \( X \)
and \( \pi : X - \pi^{-1}(B') \rightarrow \Omega' - A' - B' \) is an unramified
covering. Then, the construction of the functions \( P_f(x) \) can
be done, first for \( x' \epsilon \Omega' - A' - B' \), and, by successive
applications of the continuation theorem, extended first to
\[ x' \in \Omega' - A', \text{ then to } x' \in \Omega'. \] The rest of the proof remains the same.

**Proposition 10.** (Maximum Principle) Let \( S \) be an analytic set in \( \Omega \) and \( f \) a holomorphic function on \( \Omega \). Let \( S_a \) be irreducible, and suppose that \( f \) is not constant on \( S \) in any neighbourhood of \( a \). Then \( f(S) \) is a neighbourhood of \( f(a) \) in \( \mathbb{C} \).

**Proof.** We may suppose that \( a = 0 \), \( f(a) = 0 \). Let \( U \) be a neighbourhood of \( 0 \), and let the coordinates in \( \mathbb{C}^n \) be so chosen that \( U = U' \times U'' \), \( U' \subset \mathbb{C}^p \), \( p = \dim S_0 \) and \( \pi : S \cap U' \to U' \) is proper, has finite fibres (and satisfies Proposition 3). For \( x' \in U' \), \( \delta(x') \neq 0 \), let \( a_1, \ldots, a_k \) be the elementary symmetric functions of the values of \( f \) on \( \pi^{-1}(x') \cap S \); then the \( a_v \) admit holomorphic extensions to \( U' \). Let

\[
P(\xi; x') = \xi^k + \sum_{v=1}^{k} a_v(x')\xi^{k-v}.
\]

Then, for \( x' \in U' \), \( \delta(x') \neq 0 \), we have

\[
f(\pi^{-1}(x')) = \{ \xi \in \mathbb{C} | P(\xi, x') = 0 \}. \]

Hence, by the continuity of the roots of a polynomial, we have

\[
f(\pi^{-1}(x')) = \{ \xi \in \mathbb{C} | P(\xi, x') = 0 \} \text{ for any } x' \in U'.
\]

By assumption, we have \( f \in I(S_0) \). Hence, by Lemma 5, \( a_k(x') \neq 0 \) near \( 0 \in U' \). Further, \( a_v(0) = 0 \) for each \( v \) (since \( f(0) = 0 \)). We may suppose, after a linear change of variable in \( \mathbb{C}^p \), that \( a_k(0, \ldots, 0, x_{p-1}) \neq 0 \) near \( x = 0 \). By the preparation theorem, there exists a distinguished polynomial

\[
Q(x_p, x_1', \ldots, x_{p-1}') = x_p^m + \sum_{v=1}^{m} b_v(x_1', \ldots, x_{p-1}')x_p^{m-v}
\]

such that \( P \) and \( Q \) have the same zeros near 0. But clearly,
since $b_\mu(0) = 0$, for any $\xi$ near $0$, there is $x_p$ near $0$ with $Q(x_p, 0, \ldots, 0, \xi) = 0$; hence the set $\xi \in \mathcal{C}$ near $0$ for which there exists $x'$ near $0$ with $P(\xi, x') = 0$ is a neighbourhood of $0$. Since $f(P^{-1}(x')) = \{\xi \in \mathcal{C} \mid P(\xi, x') = 0\}$, $f(S)$ is a neighbourhood of $0$.

**Corollary 1.** A compact analytic set $S$ in $\mathbb{C}^n$ consists of a finite number of points.

**Proof.** It suffices to prove that for any holomorphic function $f$ on $\mathbb{C}^n$, $f(S)$ is finite. If it were infinite, there would exist $\alpha \in f(S) - (f(S))^O$ and $\alpha_\nu \in f(S)$, $\alpha_\nu \neq \alpha_\mu$ if $\nu \neq \mu$, such that $\alpha_\nu \rightarrow \alpha$. Let $s_\nu \in S$, $f(s_\nu) = \alpha_\nu$; by passing to a subsequence if necessary, we may suppose that $s_\nu \rightarrow s_0 \in S$, since $S$ is compact. If $S_{S_0} = \cup S_k, S_0$, then there are infinitely many $s_\nu$ on at least one $S_k, S_0$.

Hence $f$ is not constant on this component, so that $f(S)$ is a neighbourhood of $f(s_0) = \alpha$, and then $
abla f(S) - (f(S))^O$, a contradiction.

**Corollary 2.** Let $\varphi$ be a proper holomorphic map of an analytic set $S$ in an open set in $\mathbb{C}^n$ into an open set in $\mathbb{C}^m$. Then, for $y \in \mathbb{C}^m$, $\varphi^{-1}(y)$ is a finite set (being a compact analytic subset of $\mathbb{C}^n$).

**Corollary 3.** Let $\Omega$ be an open set in $\mathbb{C}^n$, $\pi : \Omega \rightarrow \mathbb{C}^p$ the projection. Let $\Omega' = \pi(\Omega)$ and $A'$ be a thin subset of $\Omega'$. If $X$ is an analytic set in $\Omega - \pi^{-1}(A')$ of dimension $p$ at any point and $\pi|X$ is proper, then $X$ is an analytic set in $\Omega$ of dimension $p$ at each point.

This follows from Corollary 2 and the remark preceding Proposition 10.

**Proposition 11.** Let $S$ be an analytic set in $\Omega$ such that $S_a$ is irreducible. Then $a$ has a fundamental system of neighbourhoods $U$ such that the set of regular points of $S$ in $U$ is connected.
Proof. Choose $U$ such that Proposition 4 is valid and further, with the notation as before, the set $S \cap \{x \in U' \mid \delta(x') \neq 0\}$ is dense in $S \cap U$. It suffices to prove that $X = S \cap \{x \in U' \mid \delta(x') \neq 0\}$ is connected. Now $\pi : X \to U' - \{x' \in U' \mid \delta(x') = 0\}$ is a finite covering. Hence if $X$ is not connected, and $Y$ is a connected component of $X$, $\pi|Y$ is also a covering. Let $h$ be the function $= 0$ on $Y$, $= 1$ on $X - Y$. If $S'$ is the set of regular points of $S$, $S' - X$ is thin, and hence $h$ has a holomorphic extension to $S'$. By the corollary to Theorem 5, there is $f \mid I(S_o)$ such that $F = fh$ is holomorphic on $S_o$. Now, for any $x' \in U'$, $\delta(x') \neq 0$, clearly there is $x \in S$ (viz $x \in Y$) with $F(x) = 0$. By Lemma 5, $F \mid I(S_o)$, so that $F$ vanishes at all points near $0$. But clearly there are points arbitrarily close to $0$ on $X - Y$ where $f \neq 0$, which implies that $F \mid I(S_o)$. This proves Proposition 11.

Corollary. If $S$, $a$ are as above, and $\dim S_a = p$, then there is a neighbourhood $V$ of such that $S$ has dimension $p$ at any point of $V \cap S$.

This follows at once from Proposition 11 and Proposition 5.

Theorem 6. Let $S$ be an analytic set in an open set $\Omega$ in $\mathbb{C}^n$ and let $0 \in S$. Suppose that $S_o$ is irreducible. Then there exists a neighbourhood $V$ of $0$ and finitely many holomorphic functions $f_1, \ldots, f_m$ in $V$ such that:

(a) the set of singular points of $S \cap V$ is precisely the set $\{x \in V \mid f_1(x) = \ldots = f_m(x) = 0\}$;

(b) each $f_i$ is a universal denominator at every point of $V$.

Proof. Choose the coordinates $(x_1, \ldots, x_n)$ in $\mathbb{C}^n$ and $U' = \{x' \in \mathbb{C}^P \mid |x'| < \epsilon'\}$, $U'' = \{x'' \in \mathbb{C}^{n-P} \mid |x''| < \epsilon''\}$ such that if $U = U' \times U''$, then the projection $\pi : U \cap S \to U'$ is proper with finite fibres.
Let $\varepsilon > 0$ be sufficiently small, and set, for $|\alpha_{ij}| < \varepsilon,$ $i = 1, \ldots, p,$ $j = 1, \ldots, n,$

$$\xi_1^{(a)} = x_1 + \sum_{j=1}^{n} \alpha_{ij} x_j.$$ 

Then, if $\varepsilon$ is small, $(\xi_1^{(a)}, \ldots, \xi_p^{(a)}, x_{p+1}, \ldots, x_n)$ are linearly independent. Let $q_1 < q'$ be fixed, and $\varepsilon$ small enough. Let $U'_1 = \{x' \in U' | |x'| < q_1\}$ and $U'_1 = U'_1 \times U''$.

Let $U_\alpha = \{x \in U | |\xi_1^{(a)}(x)| < q_1\}$, and let $\pi_\alpha : S \cap U_\alpha \to U'_1$ be the map $\pi_\alpha(x) = (\xi_1^{(a)}(x), \ldots, \xi_p^{(a)}(x))$. If $\varepsilon$ is small enough, then $\pi_\alpha$ is proper. To prove this, we remark that if $K'$ is a compact subset of $U'_1$, $\pi_\alpha^{-1}(K')$ is closed in $U$, since clearly it cannot be adherent to any point of $\partial U_\alpha$ in $U$ and is closed in $U_\alpha$. Further, if $\pi_\alpha(x) \in K'$, then $\pi(x)$ lies in a compact neighbourhood of $U'_1$ in $U$ if $\varepsilon$ is small enough; hence $\pi_\alpha^{-1}(K')$ is contained in a compact subset of $U$, and being closed in $U$, is itself compact.

Because of Corollary 1 to Proposition 11, $\pi_\alpha^{-1}(x')$ is finite for any $x' \in U'_1$.

Now, for any regular point $a$ of $S$ in $U$, there exist $\alpha_{ij}, |\alpha_{ij}| < \varepsilon,$ such that $\xi_1^{(a)}, \ldots, \xi_p^{(a)}$ form a system of coordinates at $a$. Let $W$ be a neighbourhood of $0$ such that $W \cap \bigcap_{|a_{ij}| < \varepsilon} U_\alpha$ (such a $W$ exists if $\varepsilon$ is small). If $a \in W \cap S$ and $S$ is regular at $a$, let $\alpha$ be so chosen that the $\xi^{(a)}$ form coordinates at $a$. There exists, by Theorem 5, a function $P_\alpha'$ which is a universal denominator at any point of $U_\alpha$, $P_\alpha'(a) \neq 0$, which vanishes on the singular set of $S$ in $U_\alpha$. Hence, there exists a family of holomorphic functions $\{f_t\}$ in $W$ which are universal denominators at any point of $W$ such that the singular set of $S$ in $W$ is given by $\{x \in V | f_t(x) = 0 \forall t\}$. Theorem 6 follows easily from this and Chapter II, Theorem 5 (if we replace $W$ by a smaller open set $V$).
Corollary 1. Let $S$ be an analytic set in an open set $\Omega$ in $\mathbb{C}^n$. Then the set of singular points of $S$ is an analytic set in $\Omega$.

**Proof.** Let $a \in S$ and $S_a = \cup S_{v,a}$, where the $S_{v,a}$ are the irreducible components of $S_a$. Let $S_v$ be an analytic set in a neighbourhood $W$ of $a$ inducing $S_{v,a}$. Since the $S_{v,a}$ are irreducible and none of them is contained in the union of the others, we have
$$\dim (S_{v,a} \cap S_{\mu,a}) < \min (\dim S_{v,a}, \dim S_{\mu,a})$$ if $\mu \neq v$. Hence, by the Corollary to Proposition II, we may choose $W$ so small that for $b \in S_v \cap S_{\mu} \cap W$ we have
$$\dim (S_{v,b} \cap S_{\mu,b}) < \min (\dim S_{v,b}, \dim S_{\mu,b})$$ for $\mu \neq v$.

In particular, no germ $S_{v,b}$ is contained in the union of the others. We claim that the set of singular points of $S$ which lie on $S_v$ is the union $T_v$ of the set of singular points of $S$ with $\bigcup_{\mu \neq v} (S_v \cap S_{\mu})$. In fact, it is clear that $S$ is regular at any point of $S_v$ not on $T_v$. If $b \in S_v \cap S_{\mu}$, $\mu \neq v$, then since an analytic set is clearly irreducible at regular point, $b$ is singular. If $b \notin \bigcup_{\mu \neq v} (S_v \cap S_{\mu})$, then $S_b = S_{v,b}$ so that $b$ is singular on $S$ if and only if it is singular on $S_v$. The corollary clearly follows from this and Theorem 6.

If $S_0$ is not irreducible and if $S_0 = \bigcup S_{v,0}$ is the decomposition of $S_0$ into irreducible components, let $S, S_v$ be representatives of these germs in a neighbourhood of $0$. If $f$ is a universal denominator for $S_{v,b}$ for any $b \in S_v \cap U$, and if $g$ is holomorphic on $U$, $g = 0$ on $S_{\mu}$, $\mu \neq v$, $g \neq 0$ on a dense subset of $S_v$, then $h = gf$ is a universal denominator at any point of $S \cap U$ for $S$. It follows from this that we have

**Corollary 2.** Theorem 6 remains valid if we drop the hypothesis that $S_0$ is irreducible.
Corollary 3. If $S$ is an analytic set in an open set $\Omega$ in $\mathbb{C}^n$, and $A$ is the set of singular points of $S$, and if $a \in S$, then for any $f \in \mathcal{O}_a$, which vanishes on $A_a$, there is an integer $k > 1$ such that $f^k$ is a universal for $S$ at all points near $a$.

This follows at once from Corollary 2 above and the Hilbert Nullstellensatz.

Proposition 12. Let $\Omega$ be an open set in $\mathbb{C}^n$ and $A \subset \Omega$ an analytic subset of dimension $< n - 2$. Then, for any holomorphic function on $\Omega - A$, there is a unique holomorphic function $F$ on $\Omega$ with $F|_{\Omega - A} = f$.

Proof. The uniqueness of $F$, if it exists, is obvious. Hence we have only to prove that for every $a \in A$, there is a neighbourhood $V$ and an $F$ holomorphic in $V$ with $F|_{V - A} = f|_{V - A}$. We may suppose that $a = 0$, and, after a linear change of coordinates in $\mathbb{C}^n$, that $0$ is an isolated point of the set

$$A \cap \{x_1 = \ldots = x_{n-2} = 0\}.$$

If $\varrho > 0$ is sufficiently small, and

$$a_v = (0, \ldots, 0, \frac{1}{v}, 0),$$

it is clear that, if $v$ is large, the set

$$\bar{D}_v = \{x \in \mathbb{C}^n | x_1 = \ldots = x_{n-2} = 0, x_{n-1} = \frac{1}{v}, |x_n| < \varrho\} \subset \Omega - A,$$

and

$$K_0 = \{x \in \mathbb{C}^n | x_1 = \ldots = x_{n-2} = 0, x_{n-1} = 0, |x_n| = \varrho\} \subset \Omega - A.$$

Hence, by Chapter I, Proposition 12, There is a connected neighbourhood $V$ of $0$ (containing $K_0$) and $F$ holomorphic in $V$ such that $F = f$ near $K_0$. Since $V - A$ is connected, it follows from the principle of analytic continuation that $F = f$ in $V - A$. 
Proposition 13. Let $S$ be an analytic in an open set $\Omega$ in $\mathbb{C}^n$, and let $f$ be holomorphic in $\Omega$. Suppose that, for $a \in S$, $f \neq 0$ on any irreducible component of the germ $S_a$ induced by $S$ at $a$. Then, if $S' = \{x \in S | f(x) = 0\}$, we have

$$\dim S_a' = \dim S_a - 1.$$

Proof. 1. We first suppose that $S = \Omega$ and $a = 0$. Then, by the preparation theorem, we may suppose that

$$f(x) = x^p_n + \sum_{\nu=1}^{p} a_\nu(x')x^{p-\nu}_n, \quad x' = (x_1, \ldots, x_{n-1}), \quad a_\nu(0) = 0.$$

Clearly, if $U = U' \times U''$, $U' \subset U \subset \mathbb{C}$, is a neighbourhood of 0 such that $x' \in U'$, $f(x', x_n) = 0$ imply that $x_n \in U''$, then the projection $\pi : S' = \{x \in U | f(x) = 0\} \rightarrow U'$ is surjective; hence if $g = g(x_1, \ldots, x_{n-1}) \in I(S'_0)$, then $g \equiv 0$, (since $g$ clearly vanishes on $\pi(S')$), so that, by definition of dimension, $\dim S'_0 = n - 1$.

2. For the general case, we may suppose that $a = 0$ and that $S_0$ is irreducible. If $p = \dim S_0$, we can find a neighbourhood $U = U' \times U''$ of 0, $U' \subset \mathbb{C}^p$, $U'' \subset \mathbb{C}^{n-p}$ such that $\pi : S \cap U \rightarrow U'$ is surjective, proper, and has finite fibres. Further, if $\delta$ is as in Lemma 2, $W = \{x \in S \cap U | \delta(x') \neq 0\}$ is dense in $S \cap U$. Now $\pi : W \rightarrow U' - \{x' \in U' | \delta(x') = 0\}$ is a finite covering, of say, $q$ sheets. Let, for $x' \in U'$, $\delta(x') \neq 0$, $h(x')$ be the product of the values of $f$ at the points of $S$ lying over $x'$. Then $h$ is bounded (if $U$ is small enough) and so admits a holomorphic extension, also denoted by $h$, to $U'$. Clearly $\{x' \in U', \delta(x') \neq 0, h(x') = 0\}$ is the projection of $W \cap S'$ onto $U'$. Further, if $x \in S$, $\delta(x') = 0$, we can find a sequence of points $x_v \in S$, $\delta(x_v') \neq 0$, $x_v \rightarrow x$. If $f(x) = 0$, then $f(x_v) \rightarrow 0$, hence $h(x_v') \rightarrow 0$ so that $h(x') = 0$. Hence
\( x(S') \setminus \{ x' \in U' \mid h(x') = 0 \} \), and one proves, in the same way, the converse inclusion, so that \( x(S') = \{ x' \in U' \mid h(x') = 0 \} \). Since \( f \neq 0 \) on \( S_0 \), \( h \neq 0 \). By the argument given in 1, above, we can suppose, after a linear change of coordinates in \( \mathbb{C}^p \) (and shrinking of \( U' \)) that the following holds: let \( \pi_{p-1} \) be the restriction to \( U' \) of the projection of \( \mathbb{C}^p \) onto \( \mathbb{C}^{p-1} \). Then \( T = \pi_{p-1}(x(S')) = \pi_{p-1}(U') \).

It follows since any function \( g(x_1, \ldots, x_{p-1}) \in I(S'_0) \) must vanish on \( T \), that \( I(S'_0) \cap \mathbb{C}^{p-1} = \{0\} \), so that \( \dim S'_0 = p - 1 = \dim S_0 - 1 \).

Theorem 5 has another important application.

**Theorem 7.** Let \( S \) be an analytic set in an open set in \( \mathbb{C}^n \). The space of holomorphic functions on \( S \) is complete; i.e. if \( \{f_p\} \) is a sequence of holomorphic functions on \( S \) and \( f_p \to f \), uniformly on compact subsets of \( S \), then \( f \) is holomorphic on \( S \).

**Proof.** Let \( a \in S \) and \( S_v \) be analytic sets in an open neighbourhood \( \Omega \) of \( a \) such that \( S_a = \bigcup_{v=1}^k S_{v,a} \) is the decomposition of \( S_a \) into irreducible components. By Theorem 5, applied to \( S_{v,a} \) after a linear change of coordinates in \( \mathbb{C}^n \), there is a neighbourhood \( V_v \) of \( a \) in \( \Omega \) and a holomorphic function \( u_v \) in \( V_v \) such that if \( \varphi \) is holomorphic on \( V_v \cap S \), there is a holomorphic function \( \psi_v \) in \( V_v \) such that, with \( M' > 0 \) independent of \( \varphi \), we have \( \{ x \in V_v \cap S_v \mid u_v(x) \neq 0 \} \) is dense in \( V_v \cap S_v \),

\[ u_v \varphi = \psi_v \] in \( V_v \cap S_v \),

and

\[ \| \psi_v \|_{V_v} < M' \| \varphi \|_{V_v \cap S_v} \]

If \( g_v \) is a holomorphic function in \( V_v \) vanishing on \( \bigcup_{\mu \neq v} S_{\mu} \),
while \( \{x \in S | g_v(x) \neq 0\} \) is dense in \( S_v \cap V \) (such a \( g_v \) exists, by Proposition 9' if \( V \) is small enough), then, for any \( \varphi \) holomorphic on \( V \cap S \), there is a holomorphic \( \psi_v (= g_v \psi_v') \) on \( V \) with

\[
(a) \quad \psi_v \varphi = \psi_v \text{ on } V \cap S,
\]

\[
(b) \quad \|\psi_v\|_{V_v} < M \|\varphi\|_{V \cap S}.
\]

here \( \psi_v = g_v \psi_v' \) and \( M > 0 \) is independent of \( \varphi \). Further, \( \{x \in S | V_v(x) \neq 0\} \) is dense in \( S_v \cap V \).

Let \( \mathcal{O}_a = \mathcal{O}_{\Omega_a} \) be the ring of germs of holomorphic functions on \( \Omega \) at \( a \), and \( F_a = F_a(S) \), the ideal in \( \mathcal{O}_a \) of functions vanishing on \( S_a \).

Consider the homomorphism

\[
\alpha: \mathcal{O}_a \to \mathcal{O}_a^k
\]

given by \( \alpha(f) = (v_1 f, \ldots, v_k f) \). Let \( E_a = \alpha(\mathcal{O}_a) + \sum_{\alpha} \mathcal{O}_a^k \).

By Cartan's theorem (Chapter II, Corollary 1 to Theorem 5), if a sequence \( (h_p) \) of \( k \)-tuples of holomorphic functions in a neighbourhood \( U \) of \( a \) converges uniformly on \( U \) (to \( h \) say) and \( (h_p) \in E_a \) for each \( p \), then \( (h) \in E_a \).

Let \( \{f_p\} \) be a sequence of holomorphic functions on \( \Omega \), which converges uniformly on \( \Omega \cap S \). Let \( \varphi_0 = f_0 \), \( \varphi_p = f_p - f_{p-1}, \ p > 1 \). We may suppose that

\[
\sum_{p=0}^{\infty} \|\varphi_p\|_{\Omega \cap S} < \infty.
\]

By our remarks above, there are holomorphic functions \( \psi_{p,v} (v = 1, \ldots, k) \) on \( V_v \) such that \( v v_p = \psi_{p,v} \) on \( V_v \cap S \), \( \|\psi_{p,v}\|_{V_v} < M \|\varphi\|_{V \cap S} \). If \( \psi = (\psi_1, \ldots, \psi_k) \), then \( (\alpha(\varphi_p))_a - (\psi_p)_a \in F_a \), hence \( (\psi_p)_a \in E_a \). Further, \( \sum \psi_p \) converges on \( V = \cap V_v \) since \( \sum \|\psi_p\|_V < M \sum \|\varphi_p\|_{\Omega \cap S} < \infty. \)
Let \( \psi = (\psi^{(1)}, \ldots, \psi^{(k)}) = \sum \psi_p \). Then, by our remark above, \( (\psi)_a \in \mathcal{E}_a \), so that there is a neighbourhood \( W \) of \( a \) and a holomorphic function \( \varphi \) on \( W \) such that \( v_\nu \varphi = \psi^{(\nu)} \) on \( W \cap S \).

Let \( f = \lim f_p = \sum \varphi_p \). Then, since \( v_\nu \varphi_p = \psi^{(\nu)}_p \), we have \( v_\nu f = \psi^{(\nu)} \) on \( W \cap S \), so that \( v_\nu (f - \varphi) = 0 \) on \( W \cap S \) for \( \nu = 1, \ldots, k \). But since \( \{x \in V_\nu \cap S \mid v_\nu (x) \neq 0\} \) is dense in \( V_\nu \cap S \), this implies that \( f = \varphi \) on \( W \cap S \) for each \( \nu \), so that \( \varphi = f \) on \( W \cap S \). Since \( \varphi \) is holomorphic on \( W \), this proves the theorem.

Remark. The idea of this proof is essentially that of Bungart-Rossi [7].