Let $\epsilon > 0$. Since $f$ is uniformly continuous on $\partial B$, we may find a $\delta > 0$ such that if $\xi, \zeta \in \partial B$ and $|\xi - \zeta| \leq \delta$, then $|f(\xi) - f(\zeta)| < \epsilon$. With this $\delta$ fixed, we write

$$Pf(x) - f(\zeta) = \int_{\partial B} P(x, \xi) f(\xi) \, d\sigma(\xi) - f(\zeta)$$

$$= \int_{\partial B} P(x, \xi) [f(\xi) - f(\zeta)] \, d\sigma(\xi).$$

Here we have used part 2 of Proposition 1.3.17.

But this last equals

$$\int_{\partial B \cap \{ |\xi - \zeta| \leq \delta \}} P(x, \xi) [f(\xi) - f(\zeta)] \, d\sigma(\xi)$$

$$+ \int_{\partial B \cap \{ |\xi - \zeta| > \delta \}} P(x, \xi) [f(\xi) - f(\zeta)] \, d\sigma(\xi) \equiv I + II.$$

Now $|I| \leq \epsilon \int_{\partial B} P(x, \xi) \, d\sigma(\xi) = \epsilon$ by the choice of $\delta$. On the other hand,

$$|II| \leq 2 \sup_{\partial B} |f| \int_{\partial B \cap \{ |\xi - \zeta| > \delta \}} P(x, \xi) \, d\sigma(\xi) \to 0$$

as $x \to \zeta$ by part 3 of Proposition 1.3.17. This is what we wished to prove. □

Exercises for the Reader

1. Give another proof of Theorem 1.3.18, using Theorems 1.3.10 and 1.3.12, that involves no computation. This proof will be valid for any domain $\Omega$ with $C^2$ boundary.

2. Examine the proof of Theorem 1.3.18 to see that $\lim_{r \to 1-} Pf(r\zeta)$ tends to $f(\zeta)$ uniformly in $\zeta \in \partial B$ when $f \in C(\partial B)$.

Remark: Let $\Omega \subseteq \mathbb{R}^n$ be any domain with $C^2$ boundary. It follows from the maximum principle that $G(x, y) \geq 0$. Hence, by the Hopf lemma (Exercise 22 at the end of the chapter), we conclude that $F(x, y) > 0$. Therefore, for each $x \in \Omega$, the argument in Proposition 1.3.17 shows that $\|P(x, \cdot)\|_{L^1(\partial B, d\sigma)} = 1$. Thus, for $\phi \in C(\partial \Omega)$, the functional

$$\phi \mapsto \int_{\partial B} P(x, y) \phi(y) \, d\sigma(y)$$

is bounded. From this, Theorem 1.3.12, and the maximum principle, we have the next result. □

PROPOSITION 1.3.19 The Poisson kernel for a $C^2$ domain $\Omega$ is uniquely determined by the property that it is positive and solves the Dirichlet problem for $\Delta$.

1.4 The Bergman Kernel

We have already noted that it is difficult to create an explicit integral formula, with holomorphic reproducing kernel, for holomorphic functions on a domain in $\mathbb{C}^n$. Although we shall carry out such a construction on an important class of domains in Chapter 5, we now examine one of several nonconstructive approaches to this problem. This circle of ideas, due to S. Bergman [1] and to G. Szegö [1] (some of the ideas presented here were anticipated by the thesis of S. Bochner [1]), will later be seen to have profound applications to the boundary regularity of holomorphic mappings.

In this section we will see some of the invariance properties of the Bergman kernel. This will lead to the definition of the Bergman metric (in which all biholomorphic mappings become isometries) and to such other canonical constructions as representative domains. The Bergman kernel has certain extremal properties that make it a powerful tool in the theory of partial differential equations (see S. Bergman and M. Schiffer [1]). Also, the form of the singularity of the Bergman kernel (calculable for some interesting classes of domains) explains many phenomena of the theory of several complex variables (see Chapter 8 for more on this matter).

1.4.1 General Properties of the Bergman Kernel

Let $\Omega \subseteq \mathbb{C}^n$ be a domain. Define the Bergman space

$$A^2(\Omega) = \left\{ f \text{ holomorphic on } \Omega : \int_\Omega |f(z)|^2 \, dV(z) \lesssim \|f\|_{A^2(\Omega)} < \infty \right\}.$$

LEMMA 1.4.1 Let $K \subseteq \Omega$ be compact. There is a constant $C_K > 0$, depending on $K$ and on $n$, such that

$$\sup_{z \in K} |f(z)| \leq C_K \|f\|_{A^2(\Omega)}, \quad \text{all } f \in A^2(\Omega).$$

Proof Since $K$ is compact, there is an $r(K) = r > 0$ so that for any $z \in K, B(z, r) \subseteq \Omega$. Therefore, for each $z \in K$ and $f \in A^2(\Omega)$, the first exercise following Corollary 1.3.6 implies that

$$|f(z)| = \frac{1}{V(B(z, r))} \int_{B(z, r)} f(t) \, dV(t).$$
\[ \begin{align*}
\leq \quad & (V(B(z, r)))^{-1/2} \| f \|_{L^2(B(z, r))} \\
\leq \quad & C(n)r^{-n} \| f \|_{A^2(\Omega)} \\
\equiv \quad & C_k \| f \|_{A^2(\Omega)}.  
\end{align*} \]

**Lemma 1.4.2** The space \( A^2(\Omega) \) is a Hilbert space with the inner product \( \langle f, g \rangle = \int_{\Omega} f(z)\overline{g(z)} \, dV(z) \).

**Proof** Everything is clear except for completeness. Let \( \{ f_j \} \subseteq A^2 \) be a sequence that is Cauchy in norm. Since \( L^2 \) is complete, there is an \( L^2 \) limit function \( f \). We need to see that \( f \) is holomorphic. But Lemma 1.4.1 yields that norm convergence implies normal convergence. And the last exercise in Section 1.1 or Exercise 6 at the end of Section 1.2 yields that holomorphic functions are closed under normal limits. Therefore, \( f \) is holomorphic and \( A^2(\Omega) \) is complete. \( \square \)

**Lemma 1.4.3** For each fixed \( z \in \Omega \), the functional

\[ \Phi_z : f \mapsto f(z), \quad f \in A^2(\Omega) \]

is a continuous linear functional on \( A^2(\Omega) \).

**Proof** This is immediate from Lemma 1.4.1 if we take \( K \) to be the singleton \( \{ z \} \). \( \square \)

We may now apply the Riesz representation theorem to see that there is an element \( k_z \in A^2(\Omega) \) such that the linear functional \( \Phi_z \) is represented by inner product with \( k_z \): If \( f \in A^2(\Omega) \), then for all \( z \in \Omega \) we have

\[ f(z) = (f, k_z). \]

**Definition 1.4.4** The Bergman kernel is the function \( K(z, \zeta) = k_z(\zeta) \), \( z, \zeta \in \Omega \). It has the reproducing property

\[ f(z) = \int_{\Omega} K(z, \zeta)f(\zeta) \, dV(\zeta), \quad \forall f \in A^2(\Omega). \]

**Proposition 1.4.5** The Bergman kernel \( K(z, \zeta) \) is conjugate symmetric: \( K(z, \zeta) = \overline{K(\zeta, z)} \).

**Proof** By its very definition, \( K(\zeta, \cdot) \in A^2(\Omega) \) for each fixed \( \zeta \). Therefore, the reproducing property of the Bergman kernel gives

\[ \int_{\Omega} K(z, \zeta)K(\zeta, t) \, dV(t) = \overline{K(\zeta, z)}. \]

On the other hand,

\[ \int_{\Omega} K(z, t)K(\zeta, t) \, dV(t) = \frac{\int_{\Omega} K(z, t)K(z, \t) \, dV(t)}{K(z, \zeta)} = K(z, \zeta). \]

**Proposition 1.4.6** The Bergman kernel is uniquely determined by the projections that it is an element of \( A^2(\Omega) \) in \( z \), is conjugate symmetric, and reproduces \( A^2(\Omega) \).

**Proof** Let \( K'(z, \zeta) \) be another such kernel. Then

\[ K(z, \zeta) = \overline{K(\zeta, z)} = \int K'(z, t)K(\zeta, t) \, dV(t) = \int K(z, t)K'(z, t) \, dV(t) = \overline{K'(z, \zeta)} = K'(z, \zeta). \]

Since \( L^2(\Omega) \) is a separable Hilbert space, then so is its subspace \( A^2(\Omega) \). Thus there is a complete orthonormal basis \( \{ \phi_j \}_{j=1}^{\infty} \) for \( A^2(\Omega) \).

**Proposition 1.4.7** Let \( K \) be a compact subset of \( \Omega \). Then the series

\[ \sum_{j=1}^{\infty} \phi_j(z)\overline{\phi_j(\zeta)} \]

sums uniformly on \( K \times K \) to the Bergman kernel \( K(z, \zeta) \).

**Proof** By the Riesz-Fischer and Riesz representation theorems, we obtain

\[ \sup_{z \in K} \left( \sum_{j=1}^{\infty} |\phi_j(z)|^2 \right)^{1/2} = \sup_{z \in K} \left\| \left( \sum_{j=1}^{\infty} |\phi_j(z)|^2 \right)^{1/2} \right\| = \sup_{z \in K} \left( \sum_{j=1}^{\infty} |\phi_j(z)|^2 \right)^{1/2} = \left( \sum_{j=1}^{\infty} |\phi_j(z)|^2 \right)^{1/2} \]

In the last inequality we have used Lemma 1.4.1. Therefore,

\[ \sum_{j=1}^{\infty} |\phi_j(z)|^2 \overline{|\phi_j(\zeta)|^2} \leq \left( \sum_{j=1}^{\infty} |\phi_j(z)|^2 \right)^{1/2} \left( \sum_{j=1}^{\infty} |\phi_j(\zeta)|^2 \right)^{1/2} \]

On the other hand,
and the convergence is uniform over \(z, \zeta \in K\). For fixed \(z \in \Omega\), (1.4.7.1) shows that \(\{\phi_j(z)\}_{j=1}^\infty \in L^2\). Hence we have that \(\sum \phi_j(z) \bar{\phi}_j(\zeta) \in A^2(\Omega)\) as a function of \(\zeta\). Let the sum of the series be denoted by \(K'(z, \zeta)\). Notice that \(K'\) is conjugate symmetric by its very definition. Also, for \(f \in A^2(\Omega)\), we have

\[
\int K'(z, \zeta)f(\zeta) \, dV(\zeta) = \sum f(j) \phi_j(z) = f(z),
\]

where convergence is in the Hilbert space topology. [Here \(f(j)\) is the \(j\)th Fourier coefficient of \(f\) with respect to the basis \(\{\phi_j\}\).] But Hilbert space convergence dominates pointwise convergence (Lemma 1.4.1), so

\[
f(z) = \int K'(z, \zeta)f(\zeta) \, dV(\zeta), \quad \text{all } f \in A^2(\Omega).
\]

Therefore, \(K'\) is the Bergman kernel.

\[\square\]

Remark: It is worth noting explicitly that the proof of 1.4.7 shows that

\[
\sum \phi_j(z) \bar{\phi}_j(\zeta)
\]

equals the Bergman kernel \(K(z, \zeta)\) no matter what the choice of complete orthonormal basis \(\{\phi_j\}\) for \(A^2(\Omega)\).

\[\square\]

**Proposition 1.4.10** With notation as in the definition, we have

\[
det J_{Cf} = \left| \det J_{Cf} \right|^2
\]

whenever \(f\) is a holomorphic mapping.

**Proof** We exploit the functoriality of the Jacobian. Let \(w = (w_1, \ldots, w_n) = f(z) = (f_1(z), \ldots, f_n(z))\). Write \(z_j = x_j + iy_j, w_j = \xi_j + i\eta_j, j = 1, \ldots, n\). Then by the definition of the Jacobian,

\[
d\xi_1 \wedge d\eta_1 \wedge \cdots \wedge d\xi_n \wedge d\eta_n = (\det J_{Cf}(x, y)) dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.
\]

On the other hand,

\[
d\xi_1 \wedge d\eta_1 \wedge \cdots \wedge d\xi_n \wedge d\eta_n
\]

\[
= \frac{1}{(2i)^n} d\bar{w}_1 \wedge dw_1 \wedge \cdots \wedge d\bar{w}_n \wedge dw_n
\]

\[
= \frac{1}{(2i)^n} (\det J_{Cf}(z)) (\det J_{Cf}(z)) dx_1 \wedge dz_1 \wedge \cdots \wedge dx_n \wedge dz_n
\]

\[
= (\det J_{Cf}(z))^2 dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.
\]

Equating (1.4.10.1) and (1.4.10.2) gives the result.

\[\square\]

**Exercise for the Reader**

Prove Proposition 1.4.10 using only matrix theory (no differential forms). This will give rise to a great appreciation for the theory of differential forms (see L. Bers [1] for help).

Now we can prove the holomorphic implicit function theorem.

**Theorem 1.4.11** Let \(f_j(w, z), j = 1, \ldots, m\) be holomorphic functions of \((w, z) = (w_1, \ldots, w_m, z_1, \ldots, z_n)\) near a point \((w^0, z^0) \in \mathbb{C}^m \times \mathbb{C}^n\). Assume that

\[
f_j(w^0, z^0) = 0, \quad j = 1, \ldots, m,
\]

and that

\[
\det \left( \frac{\partial f_j}{\partial w_k} \right)_{j,k=1}^m \neq 0 \quad \text{at } (w^0, z^0).
\]

Then the system of equations

\[
f_j(w, z) = 0, \quad j = 1, \ldots, m,
\]

has a unique holomorphic solution \(w(z)\) in a neighborhood of \(z^0\) that satisfies \(w(z^0) = w^0\).
Proof. We rewrite the system of equations as
\[ \text{Re} f_j(w, z) = 0, \quad \text{Im} f_j(w, z) = 0 \]
for the $2m$ real variables $\text{Re} w_k, \text{Im} w_k, k = 1, \ldots, m$. By Proposition 1.4.10 the determinant of the Jacobian over $\mathbb{R}$ of this new system is the modulus squared of the determinant of the Jacobian over $\mathbb{C}$ of the old system. By our hypothesis, this number is nonvanishing at the point $(w^0, z^0)$. Therefore, the classical implicit function theorem (see W. Rudin [1]) implies that there exist $C^1$ functions $w_k(z), k = 1, \ldots, m$, with $w(z^0) = w^0$ that solve the system. Our job is to show that these functions are, in fact, holomorphic. When properly viewed, this is purely a problem of geometric algebra.

Applying exterior differentiation to the equations
\[ 0 = f_j(w(z), z), \quad j = 1, \ldots, m, \]
yields
\[ 0 = df_j = \sum_{k=1}^{m} \frac{\partial f_j}{\partial w_k} dw_k + \sum_{k=1}^{m} \frac{\partial f_j}{\partial z_k} dz_k. \]
There are no $dz_j$'s and no $dw_k$'s because the $f_j$'s are holomorphic.

The result now follows from linear algebra only: The hypothesis on the determinant of the matrix $(\partial f_j/\partial w_k)$ implies that we can solve for $dw_k$ in terms of $dz_j$. Therefore, $w$ is a holomorphic function on $z$.

A holomorphic mapping $f : \Omega_1 \rightarrow \Omega_2$ of domains $\Omega_1 \subseteq \mathbb{C}^n, \Omega_2 \subseteq \mathbb{C}^m$ is said to be biholomorphic if it is one-to-one and onto and $\det J_z f(z) \neq 0$ for every $z \in \Omega_1$.

Exercise for the Reader
Use Theorem 1.4.11 to prove that a biholomorphic mapping has a holomorphic inverse (hence the name).

Remark: It is true, but not at all obvious, that the nonvanishing of the Jacobian determinant is a superfluous condition in the definition of “biholomorphic mapping;” that is, the non-vanishing of the Jacobian follows from the univalence of the mapping. A proof of this assertion is sketched in Exercise 37 at the end of Chapter 11.

In what follows we denote the Bergman kernel for a given domain $\Omega$ by $K_{\Omega}$.

Proposition 1.4.12 Let $\Omega_1, \Omega_2$ be domains in $\mathbb{C}^n$. Let $f : \Omega_1 \rightarrow \Omega_2$ be biholomorphic. Then
\[ \det J_z f(z) K_{\Omega_1}(f(z), f(\zeta)) \det J_{\zeta} f(\zeta) = K_{\Omega_2}(z, \zeta). \]

Proof. Let $\phi \in A^2(\Omega_1)$. Then, by change of variable,
\[
\int_{\Omega_1} \det J_z f(z) K_{\Omega_2}(f(z), f(\zeta)) \det J_{\zeta} f(\zeta) \phi(\zeta) dV(\zeta) \\
= \int_{\Omega_2} \det J_z f(z) K_{\Omega_2}(f(z), \zeta) \det J_{\zeta} f(f^{-1}(\zeta)) \phi(f^{-1}(\zeta)) \times \det J_{\zeta} f^{-1}(\zeta) dV(\zeta).
\]
By Proposition 1.4.10 this simplifies to
\[
\det J_z f(z) \int_{\Omega_2} K_{\Omega_2}(f(z), \zeta) \left( (\det J_z f(f^{-1}(\zeta)))^{-1} \phi(f^{-1}(\zeta)) \right) \, dV(\zeta).
\]
By change of variables, the expression in braces is an element of $A^2(\Omega_2)$. So the reproducing property of $K_{\Omega_2}$ applies, and the last line equals
\[
\det J_z f(z) (\det J_z f(z))^{-1} \phi(f^{-1}(f(z))) = \phi(z).
\]
By the uniqueness of the Bergman kernel, the proposition follows.

Proposition 1.4.13 For $z \in \Omega \subset \mathbb{C}^n$, it holds that $K_{\Omega}(z, z) > 0$.

Proof. Now
\[ K_{\Omega}(z, z) = \sum_{j=1}^{\infty} |\phi_j(z)|^2 \geq 0. \]
If, in fact, $K(z, z) = 0$ for some $z$, then $\phi_j(z) = 0$ for all $j$; hence $f(z) = 0$ for every $f \in A^2(\Omega)$. This is absurd.

Definition 1.4.14 For any $\Omega \subseteq \mathbb{C}^n$ we define a Hermitian metric on $\Omega$ by
\[ g_{\Omega}(z) = \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log K(z, z), \quad z \in \Omega. \]

This means that the square of the length of a tangent vector $\xi = (\xi_1, \ldots, \xi_n)$ at a point $z \in \Omega$ is given by
\[ |\xi|^2_{\Omega} = \sum_{i,j} g_{\Omega}(z) \xi_i \overline{\xi}_j. \]

The metric that we have defined is called the Bergman metric.
In a Hermitian metric \( \{ g_{ij} \} \), the length of a \( C^1 \) curve \( \gamma : [0, 1] \rightarrow \Omega \) is given by
\[
\ell(\gamma) = \int_0^1 \left( \sum_{i,j} g_{i,j}(\gamma(t)) \gamma'_i(t) \overline{\gamma'_j(t)} \right)^{1/2} \, dt.
\]

If \( P, Q \) are points of \( \Omega \), then their distance \( d_{\Omega}(P, Q) \) in the metric is defined to be the infimum of the lengths of all piecewise \( C^1 \) curves connecting the two points.

It is not a priori obvious that the Bergman metric for a bounded domain \( \Omega \) is given by a positive definite matrix at each point. We outline a proof of this fact in Exercise 39 at the end of the chapter.

**Proposition 1.4.15** Let \( \Omega_1, \Omega_2 \subseteq \mathbb{C}^n \) be domains and let \( f : \Omega_1 \rightarrow \Omega_2 \) be a biholomorphic mapping. Then \( f \) induces an isometry of Bergman metrics:
\[
||\xi||_{\Omega_2} = ||(JC_{\Omega})\xi||_{\Omega_1}
\]
for all \( z \in \Omega_1, \xi \in \mathbb{C}^n \). Equivalently, \( f \) induces an isometry of Bergman distances in the sense that
\[
d_{\Omega_2}(f(P), f(Q)) = d_{\Omega_1}(P, Q).
\]

**Proof** This is a formal exercise, but we include it for completeness:

From the definitions, it suffices to check that
\[
\sum_{i,j} g^{\Omega_2}_{i,j}(f(z)) (JC_{\Omega_2}(f(z))w_i)(\overline{JC_{\Omega_2}(f(z))w_j}) = \sum_{i,j} g^{\Omega_1}_{i,j}(z)w_iw_j
\]
for all \( z \in \Omega_1, w = (w_1, \ldots, w_n) \in \mathbb{C}^n \). But by Proposition 1.4.12,
\[
g^{\Omega_2}_{i,j}(z) = \frac{\partial^2}{\partial z_i \partial \overline{z_j}} \log K_{\Omega_2}(z, z)
\]
\[
= \frac{\partial^2}{\partial z_i \partial \overline{z_j}} \log \{|\det JC_{\Omega_2}(f(z))|^2 K_{\Omega_2}(f(z), f(z))\}
\]
\[
= \frac{\partial^2}{\partial z_i \partial \overline{z_j}} \log K_{\Omega_1}(f(z), f(z))
\]
(1.4.15.2)

since \( \log |\det JC_{f}(z)|^2 \) is locally
\[
\log (\det JC_{f}) + \log (\det JC_{f}) + C
\]

and hence is annihilated by the mixed second derivative. But (1.4.15.2) is nothing other than
\[
\sum_{\ell,m} \frac{\partial^2}{\partial z_{\ell_i} \partial \overline{z_j}} (f(z)) \frac{\partial f_{\ell_i}(z)}{\partial z_{\ell_i}} \frac{\partial f_{\ell_j}(z)}{\partial \overline{z_j}}
\]
and (1.4.15.1) follows.

**Proposition 1.4.16** Let \( \Omega \subseteq \mathbb{C}^n \) be a domain. Let \( z \in \Omega \). Then
\[
K(z, z) = \sup_{f \in \mathcal{A}^2} \frac{|f(z)|^2}{\|f\|_{\mathcal{A}^2}} = \sup_{f \in \mathcal{A}^2} \|f(z)\|^2.
\]

**Proof** Now
\[
K(z, z) = \sum |\phi_j(z)|^2
\]
\[
= \left( \sup_{\|a\|_{\mathcal{A}^2} = 1} \left| \sum a_j \phi_j(z) \right| \right)^2
\]
\[
= \sup_{\|f\|_{\mathcal{A}^2} = 1} \|f(z)\|^2,
\]
by the Riesz-Fischer theorem. This equals
\[
\sup_{f \in \mathcal{A}^2} \|f(z)\|^2.
\]

We shall use this proposition in a moment. Meanwhile, we should like to briefly mention some open problems connected with the Bergman kernel.

**The Lu Qi-Keng Conjecture**

We have already noticed that \( K_{\Omega}(z, z) > 0 \), all \( z \in \Omega \), any bounded \( \Omega \). It is reasonable to ask whether \( K_{\Omega}(z, \zeta) \) is ever equal to zero. In fact, various geometric constructions connected with the Bergman metric and associated biholomorphic invariants (which involve division by \( K' \)) make it particularly desirable that \( K \) be nonvanishing.

If \( \Omega = D \), the unit disc, then explicit calculation (which we perform below) shows that
\[
K(z, \zeta) = \frac{1}{\pi} \frac{1}{(1 - z\overline{\zeta})^2};
\]
hence, \( K(z, \zeta) \) is nonvanishing on \( D \times D \). Proposition 1.4.12 and the Riemann mapping theorem then show that the Bergman kernel for any proper, simply connected subdomain of \( \mathbb{C} \) is nonvanishing.
Chapter 1: Some Integral Formulas

1.4. The Bergman Kernel

The Bergman kernel for the annulus was studied in M. Skwarczyński [1] and was seen to vanish at some points. It is shown in N. Suita and A. Yamada [1] that if $\Omega \subseteq \mathbb{C}$ is a multiply connected domain with smooth boundary, then $K_\Omega$ must vanish—this is proved by an analysis of differentials on the Riemann surface consisting of the double of $\Omega$. By using the easy fact that the Bergman kernel for a product domain is the product of the Bergman kernels (exercise), we may conclude that any domain in $\mathbb{C}^n$ of the form $\Lambda \times \Omega$, where $\Lambda$ is multiply connected, has a Bergman kernel with zeros. The Lu Qi-Keng conjecture can be formulated as follows:

**Conjecture:** A topologically trivial domain in $\mathbb{C}^n$ has nonvanishing Bergman kernel.

It is known (R. E. Greene and S. G. Krantz [1, 2]) that a domain that is $C^\infty$ sufficiently close to the ball in $\mathbb{C}^n$ has nonvanishing Bergman kernel. Also, if a domain $\Omega$ has Bergman kernel that is bounded from zero (and satisfies a modest geometric condition), then all nearby domains have Bergman kernel that is bounded from zero. Thus it came as a bit of a surprise when H. Boas [2] showed that there exist topologically trivial domains—even ones with real analytic boundary and satisfying all reasonable additional geometric conditions—for which the Bergman kernel has zeros. See also J. Wiergerink [1], where interesting ideas contributing to the solution of this problem first arose.

**Exercise for the Reader**

The set of smoothly bounded domains for which the Lu Qi-Keng conjecture is true is closed in the Hausdorff topology on domains.

1.4.2 Smoothness to the Boundary of $K_\Omega$

It is of interest to know whether $K_\Omega$ is smooth on $\bar{\Omega} \times \Omega$. We can see from the above formula for the Bergman kernel of the disc that $K_\Omega(z, z)$ blows up as $z \to 1^-$ in the boundary. In fact this property of blowing up prevails at any boundary point of a domain at which there is a peaking function (apply Proposition 1.4.16 to a high power of the peaking function). The reference to T. Gamelin [1] contains background information on peaking functions.

However, there is strong evidence that—as long as $\Omega$ is smoothly bounded—on compact subsets of

$$\Omega \times \bar{\Omega} \backslash \left((\partial \Omega \times \partial \Omega) \cap \{z = \zeta\}\right),$$

the Bergman kernel will be smooth. For strongly convex domains (all boundary curvatures are positive), this statement is true; its proof (see N. Kerzman [3]) uses deep and powerful methods of partial differential equations.

Perhaps the most central open problem in the function theory of several complex variables is to prove that a biholomorphic mapping of two smoothly bounded domains extends to a diffeomorphism of the closures (this topic is treated in detail in Chapter 11). It is known (see S. Bell and H. Boas [1]) that a sufficient condition for this problem to have an affirmative answer on a smoothly bounded domain $\Omega \subseteq \mathbb{C}^n$ is that for any multi-index $\alpha$ there are constants $C = C_\alpha$ and $m = m_\alpha$ such that the Bergman kernel $K_\Omega$ satisfies

$$\sup_{z \in \Omega} \left| \frac{\partial^n}{\partial z^\alpha} K_\Omega(z, \zeta) \right| \leq C \cdot \delta_\Omega(\zeta)^{-m}$$

for all $\zeta \in \Omega$. Here $\delta_\Omega(w)$ denotes the distance of the point $w \in \Omega$ to the boundary of the domain.

1.4.3 Calculating the Bergman Kernel

The Bergman kernel can almost never be calculated explicitly; unless the domain $\Omega$ has a great deal of symmetry—so that a useful orthonormal basis for $A^2(\Omega)$ can be determined—there are few methods for determining $K_\Omega$.

In 1974 C. Fefferman [1] introduced a new technique for obtaining an asymptotic expansion for the Bergman kernel on a large class of domains. (For an alternate approach, see L. Boutet de Monvel and J. Sjöstrand [11].) This work enabled rather explicit estimations of the Bergman metric and opened up an entire branch of analysis on domains in $\mathbb{C}^n$ (see C. Fefferman [2], S. S. Chern and J. Moser [1], P. Klembeck [1], and R. E. Greene and S. G. Krantz [11], for example).

The Bergman theory that we have presented here would be a bit hollow if we did not at least calculate the kernel in a few instances. We complete the section by addressing that task.

Restrict attention to the ball $B \subseteq \mathbb{C}^n$. The functions $z^\alpha$, $\alpha$ a multi-index are each in $A^2(B)$ and are pairwise orthogonal by the symmetry of the ball. By the uniqueness of the power series expansion for an element of $A^2(B)$, the elements $z^\alpha$ form a complete orthonormal system on $B$ (their closed linear span is $A^2(B)$). Setting

$$\gamma_\alpha = \int_B |z^\alpha|^2 dV(z),$$

we see that $\{z^\alpha/\sqrt{\gamma_\alpha}\}$ is a complete orthonormal system in $A^2(B)$. Thus, by Proposition 1.4.7,

$$K(z, \zeta) = \sum_\alpha \frac{z^\alpha \bar{\zeta}^\alpha}{\gamma_\alpha}.$$

If we want to calculate the Bergman kernel for the ball in closed form, we need to calculate the $\gamma_\alpha$'s. This requires some lemmas from real analysis. These lemmas will be formulated and proved on $\mathbb{R}^N$ and $B_N = \{x \in \mathbb{R}^N : |x| < 1\}$. 

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1.4. The Bergman Kernel

**Lemma 1.4.17** We have that
\[ \int_{\mathbb{R}^n} e^{-\pi|z|^2} \, dz = 1. \]

**Proof** The case \( N = 1 \) is familiar from calculus (or see E. M. Stein and G. Weiss [1]). For the \( N \)-dimensional case, write
\[ \int_{\mathbb{R}^n} e^{-\pi|z|^2} \, dz = \prod_{j=1}^{N} \int_{\mathbb{R}} e^{-\pi x_j^2} \, dx_j \]
and apply the one-dimensional result. \( \Box \)

Let \( \sigma \) be the unique rotationally invariant area measure on \( S_{N-1} = \partial B_N \) (see Appendix II) and let \( \omega_{N-1} = \sigma(\partial B) \).

**Lemma 1.4.18** We have
\[ \omega_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)}, \]
where
\[ \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \]
is Euler’s gamma function.

**Proof** Introducing polar coordinates we have
\[ 1 = \int_{\mathbb{R}^n} e^{-\pi|z|^2} \, dz = \int_{S^{N-1}} d\sigma \int_0^\infty e^{-\pi r^2} r^{N-1} \, dr \]
or
\[ \frac{1}{\omega_{N-1}} = \int_0^\infty e^{-\pi r^2} r^N \, dr. \]

Letting \( s = r^2 \) in this last integral and doing some obvious manipulations yields the result. \( \Box \)

Now we return to \( B \subseteq \mathbb{C}^n \). We set
\[ \eta(k) = \int_{\partial B} |z_1|^{2k} \, d\sigma, \quad N(k) = \int_{B} |z_1|^{2k} \, dV(z), \quad k = 0, 1, \ldots. \]

**Lemma 1.4.19** We have
\[ \eta(k) = \pi^n \frac{2(k!)^n}{(k + n - 1)!}, \quad N(k) = \pi^n \frac{k!}{(k + n)!}. \]

**Proof** Polar coordinates show easily that \( \eta(k) = 2(k+n)N(k) \). So it is enough to calculate \( N(k) \). Let \( z = (z_1, z_2, \ldots, z_n) = (z_1, z') \). We write
\[ N(k) = \int_{|z_1| < 1} |z_1|^{2k} \, dV(z) \]
\[ = \int_{|z| < 1} \left( \int_{|z_1| < \sqrt{1-|z|^2}} |z_1|^{2k} \, dV(z_1) \right) dV(z') \]
\[ = 2\pi \int_{|z'| < 1} \int_0^{\sqrt{1-|z'|^2}} r^{2k} \, dr \, dV(z') \]
\[ = 2\pi \int_{|z'| < 1} \left( \frac{1 - |z'|^2} {2k + 2} \right) dV(z') \]
\[ = \frac{\pi \omega_{2n-3}} {k+1} \int_0^1 (1 - s)^{k+1} s^{n-3} \, ds \]
\[ = \frac{\pi \omega_{2n-3}} {k+1} \int_0^1 (1 - s)^{k+1} s^{n-1} \, ds \]
\[ = \frac{\pi^{n+1}} {k+1} \omega_{2n-3} \beta(n - 1, k + 2), \]
where \( \beta \) is the classical beta function of special function theory (see G. Carrier, M. Crook, and C. Pearson [1] or E. Whittaker and G. Watson [1]). By a standard identity for the beta function, we then have
\[ N(k) = \frac{\pi \omega_{2n-3}} {2(k+1)} \frac{\Gamma(n-1)\Gamma(k+2)} {\Gamma(n+k+1)} \]
\[ = \frac{2\pi^{n-1}} {2(k+1)} \frac{\Gamma(n-1)\Gamma(k+2)} {\Gamma(n+k+1)} \]
\[ = \frac{\pi^n k!} {(k+n)!}. \]

This is the desired result. \( \Box \)

**Lemma 1.4.20** Let \( z \in B \subseteq \mathbb{C}^n \) and \( 0 < r < 1 \). The symbol \( \mathbf{1} \) denotes the point \((1,0,\ldots,0)\). Then
\[ K_B(z, r\mathbf{1}) = \frac{n!}{\pi^n} \frac{1} {(1-rz_1)^{n+1}}. \]

**Proof** Refer to the formula preceding Lemma 1.4.17. Then
\[ K_B(z, r\mathbf{1}) = \sum_\alpha \frac{z_\alpha (r\mathbf{1})^\alpha} {\gamma_\alpha} = \sum_{k=0}^\infty \frac{z_1^k r^k} {N(k)} \]
This is the desired result. \hfill \Box

**OREM 1.4.21** If \( z, \zeta \in B \), then

\[
K_B(z, \zeta) = \frac{n!}{\pi^n (1 - z \cdot \zeta)^{n+1}},
\]

where \( z \cdot \zeta = z_1 \zeta_1 + z_2 \zeta_2 + \cdots + z_n \zeta_n \).

**Proof** Let \( z = r \hat{z} \in B \), where \( r = |z| \) and \( |\hat{z}| = 1 \). Also, fix \( \zeta \in B \). Choose a unitary rotation \( \rho \) such that \( \rho \hat{z} = 1 \). Then, by Proposition 1.4.12 and Lemma 1.4.20 we have

\[
K_B(z, \zeta) = K_B(r \hat{z}, \zeta) = K(r \rho^{-1} 1, \zeta) = K(\rho \zeta, r 1) = \frac{n!}{\pi^n (1 - r (\rho \zeta) \cdot 1)^{n+1}}.
\]

**PROPOSITION 1.4.22** The Bergman metric for the ball \( B = B(0, 1) \subseteq \mathbb{C}^n \) is given by

\[
g_{ij}(z) = \frac{n + 1}{(1 - |z|^2)^2} \left[ (1 - |z|^2) \delta_{ij} + z_i \bar{z}_j \right].
\]

**Proof** Since \( K(z, z) = n!/(\pi^n (1 - |z|^2)^{n+1}) \), this is a routine computation that we leave to the reader. \hfill \Box

**COROLLARY 1.4.23** The Bergman metric for the disc (i.e., the ball in dimension one) is

\[
g_{ij}(z) = \frac{2}{(1 - |z|^2)^2}, \quad i = j = 1.
\]

This is the well-known Poincaré, or Poincaré-Bergman, metric.

**PROPOSITION 1.4.24** The Bergman kernel for the polydisc \( D^n(0, 1) \subseteq \mathbb{C}^n \) is the product

\[
K(z, \zeta) = \frac{1}{\pi^n} \prod_{j=1}^n \frac{1}{(1 - z_j \zeta_j)^2}.
\]

**Proof** This is left as an exercise for the reader. Use the uniqueness property of the Bergman kernel. \hfill \Box

**Exercise for the Reader**

Calculate the Bergman metric for the polydisc.

**1.4.4 The Poincaré-Bergman Metric on the Disc**

If \( D \subseteq \mathbb{C} \) is the unit disc, \( z \in D \), then Corollary 1.4.23 shows that

\[
|w|_{B,z} = \left( \frac{2|w|^2}{(1 - |z|^2)^2} \right)^{1/2} = \frac{\sqrt{2} |w|}{1 - |z|^2},
\]

where the subscript \( B \) indicates that we are working in the Bergman metric. We now use this formula to derive an explicit expression for the Poincaré distance from \( 0 \in D \) to \( r + i0 \in D, 0 < r < 1 \). Call this distance \( d(0, r) \). Then

\[
d(0, r) = \inf \left\{ \int_0^1 |\gamma'(t)|_{B, \gamma(t)} \, dt : \gamma \text{ is a curve in } D, \gamma(0) = 0, \gamma(1) = r + i0 \right\}.
\]

Elementary comparisons show that, among curves of the form \( \psi(t) = rt + tu(t), 0 \leq t \leq 1 \), the curve \( \gamma(t) = tr + i0 \) is the shortest in the Poincaré metric. Further elementary arguments show that a general curve of the form \( \psi(t) = vt + u(t) \) is always longer than some corresponding curve of the form \( rt + tu(t) \).

We leave the details of these assertions to the reader. Thus

\[
d(0, r) = \int_0^1 \frac{\sqrt{2} |r|}{(1 - (rt)^2)} \, dt = \sqrt{2} \int_0^r \frac{1}{1 - t^2} \, dt = \frac{1}{\sqrt{2}} \log \left( \frac{1 + r}{1 - r} \right).
\]
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Since rotations are conformal maps of the disc, we may next conclude that
\[
d(0, re^{i\theta}) = \frac{1}{\sqrt{2}} \log \left( \frac{1 + r}{1 - r} \right).
\]

Finally, if \(w_1, w_2\) are arbitrary, then the Möbius transformation
\[
\phi: z \mapsto \frac{z - w_1}{1 - \overline{w}_1 z}
\]
satisfies \(\phi(w_1) = 0, \phi(w_2) = (w_2 - w_1)/(1 - \overline{w}_1 w_2)\). Then Proposition 1.4.15 yields that
\[
d(w_1, w_2) = d \left( 0, \frac{w_2 - w_1}{1 - \overline{w}_1 w_2} \right) = \frac{1}{\sqrt{2}} \log \left( \frac{1 + \frac{w_2 - w_1}{1 - \overline{w}_1 w_2}}{1 - \frac{w_2 - w_1}{1 - \overline{w}_1 w_2}} \right).
\]

We note in passing that the expression \(\rho(w_1, w_2) = |(w_1 - w_2)/(1 - \overline{w}_1 w_2)|\) is called the pseudohyperbolic distance. It is also conformally invariant, but it does not arise from integrating an infinitesimal metric (i.e., lengths of tangent vectors at a point). A fuller discussion of both the Poincaré metric and the pseudohyperbolic metric on the disc may be found in J. Garnett [1] and in S. Krantz [20].

1.4.5 Appendix to Section 1.4: The Biholomorphic Inequality of the Ball and the Polydisc

**Theorem 1.4.25** There is no biholomorphic map \(\phi\) of the bidisc \(D^2(0,1)\) to the ball \(B(0,1) \subseteq \mathbb{C}^2\).

**Proof** Suppose, seeking a contradiction, that there is such a map. Since Möbius transformations act transitively on the disc, pairs of them act transitively on the bidisc. Therefore, we may compose \(\phi\) with a self-map of the bidisc and assume that \(\phi\) maps 0 to 0.

If \(Y \in \partial B\), then the disc \(d_Y = \{ z \in B : z = \zeta Y, \zeta \in \mathbb{C}, |\zeta| < 1 \}\) is a totally geodesic submanifold of \(B\) (informally, this means that if \(P, Q\) are points of \(d_Y\), then the geodesic connecting them in the Riemannian manifold \(d_Y\) is the same as the geodesic connecting them in the Riemannian manifold \(B\)—see S. Kobayashi and K.Nomizu [1]). Here we work in the Bergman metric.

By our discussion in the calculation of the Poincaré metric, we may conclude that the geodesics, or paths of least length, emanating from the origin in the ball are the rays \(r\gamma: t \mapsto t Y\). (This assertion may also be derived from symmetry considerations.)

Likewise, if \(\alpha, \beta \in \mathbb{C}, |\alpha| = 1, |\beta| = 1\), then the disc \(D^\alpha = \{ (\zeta \alpha, \zeta \beta) : \zeta \in D \} \subseteq D^2(0,1)\) is a totally geodesic submanifold of \(D^2(0,1)\). Again we may apply our discussion of the Poincaré metric on the disc to conclude that the geodesic curve emanating from the origin in the bidisc in the direction \(X = (\alpha, \beta)\) is \(\psi_{\alpha\beta}: t \mapsto t X\). A similar argument shows that the curve \(t \mapsto (t, 0)\) is a geodesic in the bidisc.

Now if \(t \mapsto t X\) is one of the above-mentioned geodesics on the bidisc, then it will be mapped under \(\phi\) to a geodesic \(t \mapsto t Y\) in the ball. If \(0 < t_1 < t_2 < 1\), then the points \(t_1 X, t_2 X \in D^2\) will be mapped to points \(t_1 Y, t_2 Y \in B\) and it must be that \(0 < t_1' < t_2' < 1\), since \(\phi\) is an isometry and hence must map the point \(t_2 X\) to a point further from the origin than it maps \(t_1 X\) (because \(t_2 X\) is further from the origin than \(t_1 X\)). It follows that the limit
\[
\lim_{t \to 1^-} \phi(t X)
\]
exists for every choice of \(X\) and the limit lies in \(\partial B\). After composing \(\phi\) with a rotation we may suppose that \(\{ \phi(t(1, 0)) \}\) terminates at \((1, 0)\).

Now consider the function \(f(x_1, x_2) = (x_1 + 1)/2\) on \(B\). This function has the property that \(f(1, 0) = 1, f\) is holomorphic on a neighborhood of \(B\), and \(|f(z)| < 1\) for \(z \in B \setminus \{(1, 0)\}\). For \(0 < r < 1\) we invoke the mean value property for a harmonic function to write
\[
\frac{1}{2\pi} \int_0^{2\pi} f \circ \phi(r, r e^{i\theta}) d\theta = f \circ \phi(r, 0).
\]

As \(r \to 1^-\) the right-hand side tends to \(\lim_{r \to 1^-} f(t, 0) = 1\). However, each of the paths \(r \to (r, r e^{i\theta})\) is a geodesic in the bidisc, as discussed above, and for different \(\theta \in [0, 2\pi]\) they are distinct. Thus the curves \(r \to \phi(r, r e^{i\theta})\) have distinct limits in \(\partial B\), and these limits will be different from the point \((1, 0) \in \partial B\). In particular, \(\lim_{r \to 1^-} f \circ \phi(r, r e^{i\theta})\) exists for each \(\theta \in [0, 2\pi]\) and assumes a value of modulus strictly less than 1.

By the Lebesgue-dominated convergence theorem, we may pass to the limit as \(r \to 1^-\) in the left side of (1.4.25.1) to obtain a limit that must be strictly less than one in absolute value. That is the required contradiction.\]

1.5 The Szegö and Poisson-Szegö Kernels

The basic theory of the Szegö kernel is similar to that for the Bergman kernel—they are both special cases of a general theory of "Hilbert spaces with reproducing
The Szegő kernel may be thought of as representing a map

$$S : f \mapsto \int_{\partial \Omega} f(\zeta) S(\cdot, \zeta) \, d\sigma(\zeta)$$

from $L^2(\partial \Omega)$ to $H^2(\partial \Omega)$. Since $S = S'$ is self-adjoint and idempotent, it is the Hilbert space projection of $L^2(\partial \Omega)$ to $H^2(\partial \Omega)$.

The Poisson-Szegő kernel is obtained by a formal procedure from the Szegő kernel: This procedure manufactures a positive reproducing kernel from one that is not necessarily positive. Note in passing that, just as we argued for the Bergman kernel in the last section, $S(z, z)$ is never 0 when $z \in \Omega$.

**PROPOSITION 1.5.1** Define

$$P(z, \zeta) = \frac{|S(z, \zeta)|^2}{S(z, z)}, \quad z \in \Omega, \ \zeta \in \partial \Omega.$$

Then for any $f \in A(\Omega)$ and $z \in \Omega$, it holds that

$$f(z) = \int_{\partial \Omega} f(\zeta) P(z, \zeta) \, d\sigma(\zeta).$$

**Proof** Fix $z \in \Omega$ and $f \in A(\Omega)$ and define

$$u(\zeta) = f(\zeta) \frac{S(z, \zeta)}{S(z, z)}, \quad \zeta \in \partial \Omega.$$

Then $u \in H^2(\partial \Omega)$; hence

$$f(z) = u(z) = \int_{\partial \Omega} S(z, \zeta) u(\zeta) \, d\sigma(\zeta) = \int_{\partial \Omega} P(z, \zeta) f(\zeta) \, d\sigma(\zeta).$$

This is the desired formula. □

**Remark:** In passing to the Poisson-Szegő kernel we gain the advantage of positivity of the kernel (for more on this circle of ideas, see Chapter 8 and also Chapter 1 of Y. Katznelson [1]). However, we lose something in that $P(z, \zeta)$ is no longer holomorphic in the $z$ variable and not holomorphic in the $\zeta$ variable. The literature on this kernel is rather sparse and there are many unresolved questions. □

As an exercise, use the paradigm of Proposition 1.5.1 to construct a positive kernel from the Cauchy kernel on the disc (be sure to first change notation in
the usual Cauchy formula so that it is written in terms of arc length measure on the boundary). What familiar kernel results?

Like the Bergman kernel, the Szegö and Poisson-Szegö kernels can almost never be explicitly computed. They can be calculated asymptotically in a number of important instances, however (see C. Fefferman [1] and L. Boutet de Monvel and J. Sjöstrand [1])). We will give explicit formulas for these kernels on the ball. The computations are similar in spirit to those in Section 1.4; fortunately, we may capitalize on much of the work done there.

**Lemma 1.5.5** The functions \( \{ z^n \} \), where \( \alpha \) ranges over multi-indices, are pairwise orthogonal and span \( H^2(\mathbb{D}) \).

**Proof** The orthogonality follows from symmetry considerations. For the completeness, notice that it suffices to see that the span of \( \{ z^n \} \) is dense in \( A(B) \) in the uniform topology on the boundary. By the Stone-Weierstrass theorem, the closed algebra generated by \( \{ z^n \} \) is all of \( C(\partial B) \). But the monomials \( \bar{z}^n, \alpha \neq 0 \), are orthogonal to \( A(B) \) (use the power series expansion about the origin to see this). The claimed density follows.

**Lemma 1.5.6** Let \( \mathbf{1} = (1, 0, \ldots, 0) \). Then

\[
S(z, \mathbf{1}) = \frac{(n-1)!}{2\pi^n} \frac{1}{(1-z)^n}.
\]

**Proof** We have that

\[
S(z, \mathbf{1}) = \sum_{\alpha} \frac{z^\alpha \cdot \mathbf{1}^\alpha}{\|z^\alpha\|^2_{L^2(\partial B)}} = \sum_{k=0}^{\infty} \frac{z_k}{2\pi^n} \sum_{k=0}^{\infty} z_k \frac{(k+n-1)!}{k!} = \frac{(n-1)!}{2\pi^n} \sum_{k=0}^{\infty} \left( \frac{k+n-1}{n-1} \right) z_k = \frac{(n-1)!}{2\pi^n} \frac{1}{(1-z)^n}.
\]

**Corollary 1.5.6** The Poisson-Szegö kernel for the ball is

\[
S(z, \zeta) = \frac{(n-1)!}{2\pi^n} \frac{1}{1 - z \cdot \zeta^n}.
\]

**Exercise for the Reader**
Calculate the Szegö and Poisson-Szegö kernels for the polydisc.

### 1.6 Afterword to Chapter 1

It follows from work of A. Gleason [2] and L. Bungart [1] that Cauchy-type integral formulas that are holomorphic in the \( z \) variable (this is crucial for constructing holomorphic functions) exist on virtually any domain. However these references tell us almost nothing about the form of the integral kernel—whether its singularity is only on the diagonal or at what rate it blows up. The Bergman and Szegö theories give other methods for producing integral formulas on a wide class of domains.

Thus we have several "canonical kernels" on domains in \( \mathbb{C}^n \): the Poisson kernel, the Bergman kernel, and the Szegö kernel. Unfortunately, these kernels are virtually never computable (however see C. Fefferman [1], L. Boutet de Monvel and J. Sjöstrand [1], and L. Hun [1]). Later on we shall learn of a construction by G. M. Henkin (variants were also constructed independently by E. Ramirez [1] and H. Grauert and I. Lieb [1]) that yields rather explicitly computable kernels on a large class of domains in \( \mathbb{C}^n \). For many practical applications, the Henkin kernel is just as useful as the Szegö kernel. And it turns out that the Szegö kernel may be expressed as an asymptotic expansion in terms of powers of the Henkin kernel (see N. Kerzman and E. M. Stein [2]).

The connection between canonical kernels and computable kernels has opened up new vistas and has made accessible many deep results in function theory and geometry (see N. Kerzman and E. M. Stein [1, 2], R. E. Greene and S. G. Krantz [1-11], D. R. Phong and E. M. Stein [1], C. Fefferman [1, 2], L. Boutet de Monvel and J. Sjöstrand [1], and S. Bell [3]). The subject of integral representations will be a fertile area of research for some time to come.

### Exercises

1. Prove that if \( f \) and \( f^2 \) are real-valued harmonic on a domain \( \Omega \subseteq \mathbb{R}^N \), then \( f \) is constant. What can you say for complex-valued \( f \)?