ON THE REMOVAL OF SINGULARITIES OF ANALYTIC SETS

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1. INTRODUCTION

The purpose of this paper is to give a measure-theoretic condition guaranteeing the removability of singularities of an analytic set. Our main theorem, which states this condition, generalizes a result of Remmert and Stein [8] and is similar to a recent result of Bishop [2].

We write R^n and C^n for real and complex Euclidean n-space. By an *analytic set* in an open subset U of C^n we mean a closed analytic subvariety of U; by an *analytic function* on U we mean a complex-valued function holomorphic on U. Suppose E is a closed, nowhere dense subset of U, and A is an analytic set in U - E. In analogy with the theory of analytic functions, we can regard the points of E as "singularities" of A and ask whether these "singularities" are "essential" or "removable." Our main result is the following theorem on the removability of singularities.

THEOREM. *Let* U *be open in* C^n, *and let* E *be closed in* U. *Let* A *be a pure* k-*dimensional analytic set in* U - E, *and let* A' *be the closure of* A *in* U. *If* E *has Hausdorff* (2k - 1)-*measure zero, then* A' *is a pure* k-*dimensional analytic set in* U.

In Section 3, we also prove some results concerning the removability of singularities of analytic functions that we apply in the proof of our theorem.

In this paragraph, let U, E, A, and A' be given as above. We let H^p denote Hausdorff p-measure, which we discuss in Section 2. Remmert and Stein proved in 1953 that if E is analytic in U and of dimension less than k, then A' must be analytic in U [8, Satz 13, p. 299]. This result is often referred to as the Remmert-Stein Theorem, although it is only a special case of the main result of [8]. Our theorem generalizes this result of Remmert and Stein, since if E is analytic of dimension less than k, then $H^{2k-1}(E) = 0$ (see Section 2). In fact, our theorem tells us that A' is analytic whenever E is a countable union of real $\mathscr{C}^1$-submanifolds of (real) dimension at most 2k - 2. Bishop proved a similar generalization of this result of Remmert and Stein, which states that if $H^{2k}(E) = 0$ and E is contained in an analytic subset of U that does not intersect A, then A' must be analytic [2, Lemma 9, p. 294]. In Section 4, we use the methods of Bishop to complete the proof of our theorem. We also give a proof (Section 5) of an integral-geometric inequality that is stated without proof in [2].

An equivalent statement of our theorem is the following characterization of analytic sets:

Let U be open in C^n. Then a subset A of U is an analytic set of pure dimension k if and only if A is the closure (in U) of some complex k-dimensional submanifold M of U such that

$$H^{2k-1}(A - M) = 0.$$

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(Note that M is not necessarily closed in U, but that it must be relatively closed.) This result follows from our theorem if we let $E = A - M$; conversely, our theorem follows from this characterization of analytic sets if we let M equal the set of regular (simple) points of A. The terminology and basic results concerning analytic sets that we use in this paper can be found in [5 (see especially Chapter III)].

2. HAUSDORFF MEASURE

To prove our theorem, we need some elementary results involving Hausdorff measures. We state these results in greater generality than is necessary for the purpose of this paper, since the proofs are most apparent in the general setting. For a more detailed discussion of similar results, the reader can consult the literature of the foundations of integral geometry—for example [4]. We begin by introducing our terminology and notation.

Definition. Let μ be a nonnegative measure on X. For any function $f: X \to [0, +\infty]$, the *upper integral* of f is defined as

$$\int^* f\, d\mu = \inf \left\{ \int g\, d\mu : \text{ g is integrable and } g \geq f \right\}.$$

By convention (throughout this paper), $\inf \emptyset = +\infty$.

We state the following elementary properties of the upper integral:

1) If $f: X \to [0, +\infty]$ is such that $\int^* f\, d\mu = 0$, then $f = 0$ almost everywhere.

Let $f_n: X \to [0, +\infty]$ for $n = 1, 2, \cdots$. Then

2) $\int^* \liminf f_n\, d\mu \leq \liminf \int^* f_n\, d\mu$,

3) $\int^* \sum f_n\, d\mu \leq \sum \int^* f_n\, d\mu$.

Definition. Let A be a subset of a metric space X. Let $\delta(A)$ denote the diameter of A, and let

$$\delta^p(A) = [\delta(A)]^p \text{ for } p > 0,$$

$$\delta^0(A) = \begin{cases} 1 & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset. \end{cases}$$

For $p \geq 0$ and $\varepsilon > 0$, define

$$H^p_\varepsilon(A) = \inf \left\{ \sum_1^\infty \delta^p(A_n) : A \subset \bigcup A_n \text{ and } \delta(A_n) < \varepsilon \right\},$$

$$H^p(A) = \lim_{\varepsilon \to 0^+} H^p_\varepsilon(A) = \sup_{\varepsilon > 0} H^p_\varepsilon(A).$$

We refer to H^p as *Hausdorff* p-*measure*. For any $A \subset X$, $H^0(A)$ equals the number of points of A. These set functions H^p (for $p \geq 0$) are regular metric outer measures, and hence the Borel sets are H^p-measurable (see, for instance, [7]). If A is H^p-σ-finite, then $H^r(A) = 0$ whenever $r > p$. If M is an n-dimensional Riemannian manifold, then M is H^n-σ-finite, the compact sets in M have finite Hausdorff n-measure, and the open sets have nonzero Hausdorff n-measure. (For a nonnegative integer k, the usual notion of k-volume in a Riemannian manifold differs from H^k by a universal multiplicative constant

$$c_k = \text{Volume}\left\{x \in R^k\colon \|x\| \le \frac{1}{2}\right\} = \frac{\pi^{k/2}}{2^k \Gamma(k/2+1)}.$$

See [9], for instance. Many authors refer to $c_k H^k$ as Hausdorff k-measure.) It follows that k-dimensional analytic sets have Hausdorff $(2k+1)$-measure zero, since a k-dimensional analytic set is the union of manifolds of real dimension at most 2k. One can show that the Hausdorff 2k-measure of a k-dimensional analytic set is locally finite (see [10, pp. 13-15]), but we do not need this result.

If $f\colon X \to Y$ is a mapping of metric spaces, we write

$$\text{Lip}(f) = \sup_{a \neq b} \frac{d(f(a), f(b))}{d(a, b)} \le +\infty,$$

and we say that f satisfies a Lipschitz condition of order 1 ($f \in L_1$) if $\text{Lip}(f) < +\infty$. A subset A of a metric space X is said to be of type $L(p, c)$, for $0 \le c < +\infty$, if there exists $\delta > 0$ such that $H^p(B) \le c\,\delta^p(B)$ for all $B \subset A$ with diameter smaller than δ.

Each compact, n-dimensional Riemannian manifold is clearly of type $L(n, c)$ for some constant c. It can be shown that R^n is of type $L(n, 1)$ (see [9]), and hence each compact Riemannian manifold is of type $L(n, 1+\varepsilon)$ for all positive ε.

Let $\tilde{H}^p$ denote the restriction of H^p to the Borel sets, and write

$$\int^* f(x)\,d^p x = \int^* f(x)\,d\tilde{H}^p(x) = \int^* f\,d\tilde{H}^p.$$

The following fundamental inequality is our basic measure-theoretic tool.

LEMMA 1 (Federer). *Let* X *and* Y *be metric spaces, and let* f *map* X *into* Y, *with* $\text{Lip}(f) = \lambda < +\infty$. *Let* $\alpha \ge 0$, $\beta > 0$, *and suppose that* f(X) *is of type* $L(\beta, c)$ *and* $H^{\alpha+\beta}(X) < +\infty$. *Then*

$$\int^* H^\alpha(f^{-1}(y))\,d^\beta y \le c\lambda^\beta H^{\alpha+\beta}(X).$$

Federer proved a more general version of Lemma 1 [4, p. 243]. We give an elementary proof of our version: Pick δ as in the definition of $L(\beta, c)$. Let $A \subset X$ with $\delta(A) < \delta/(\lambda+1)$, and let $B = f(A)$. Then $\delta(\overline{B}) \le \lambda\delta(A) < \delta$, and therefore

$$\int^* \delta^\alpha(A \cap f^{-1}(y))\,d^\beta y \le \delta^\alpha(A)\tilde{H}^\beta(\overline{B}) \le c\lambda^\beta \delta^{\alpha+\beta}(A).$$

Let $\varepsilon < \delta/(\lambda+1)$, and pick a covering $\{A_n\}$ of X such that

$$\delta(A_n) < \varepsilon \quad \text{and} \quad \sum \delta^{\alpha+\beta}(A_n) \le H_\varepsilon^{\alpha+\beta}(X) + \varepsilon.$$

Then, by the above inequality and property 3 of the upper integral,

$$\int^{*} H^{\alpha}_{\varepsilon}(f^{-1}(y))\,d^{\beta}y \le c\lambda^{\beta}\sum \delta^{\alpha+\beta}(A_n) \le c\lambda^{\beta}H^{\alpha+\beta}_{\varepsilon}(X) + c\lambda^{\beta}\varepsilon,$$

and therefore

$$\int^{*} H^{\alpha}(f^{-1}(y))\,d^{\beta}y \le \liminf \int^{*} H^{\alpha}_{1/n}(f^{-1}(y))\,d^{\beta}y \le c\lambda^{\beta}H^{\alpha+\beta}(X).$$

COROLLARY 1. *Let* X *be a metric space, and let* $a \in X$. *Then*

$$\int^{*}_{[0,+\infty)} H^{\alpha}(S_r(a))\,d^{1}r \le H^{\alpha+1}(X) \qquad \text{for } \alpha \ge 0,$$

where $S_r(a) = \{x \in X\colon d(x, a) = r\}$.

Proof. Consider the function $f\colon X \to [0, +\infty)$ given by $f(x) = d(x, a)$.

COROLLARY 2. *Let* $H^1(X) = 0$, *and let* $a \in X$. *Then* $S_r(a)$ *is empty for* (H^1)-*almost all* r.

COROLLARY 3. *If* $H^{n+1}(X) = 0$, *then the topological dimension of* X *is at most* n.

Our proof of Lemma 1 was inspired by Hurewicz and Wallman's proof [6, pp. 104-105] of Corollary 3.

COROLLARY 4. *Let* A *be an arbitrary subset of* R^n, *let* $\alpha \ge 0$, *and let* $\pi_k\colon R^n \to R^k$ *denote the projection onto the first* k *coordinates.*

(i) *If* $H^{k+\alpha}(A) = 0$, *then* $H^{\alpha}(A \cap \pi_k^{-1}(x)) = 0$ *for* (H^k)-*almost all* $x \in R^k$.

(ii) *If* $H^{k+\alpha}(A) < +\infty$, *then* $H^{\alpha}(A \cap \pi_k^{-1}(x)) < +\infty$ *for* (H^k)-*almost all* $x \in R^k$.

LEMMA 2. *Let* Y *be an arbitrary subset of* C^n *with* $0 \notin Y$, *and let* $\alpha \ge 0$. *If* $H^{2k+\alpha}(Y) = 0$, *then there exists a complex* (n - k)-*plane* P *through* 0 *such that*

$$H^{\alpha}(Y \cap P) = 0.$$

Remark. If $\alpha > 0$, the hypothesis $0 \notin Y$ is inessential. If $\alpha = 0$, the conclusion asserts that $Y \cap P = \emptyset$.

Proof. We use downward induction on k. If k = n, the lemma is trivial. Now suppose the lemma is true for k + 1, and let Y be given as above. Then the Hausdorff $(2k + 2 + \alpha)$-measure of Y also vanishes; therefore we can pick a complex (n - k - 1)-plane Q through 0 such that $H^{\alpha}(Y \cap Q) = 0$. After rotating the coordinate system, we may assume that

$$Q = \{z_1 = \cdots = z_{k+1} = 0\}.$$

Let

$$\pi\colon C^n - Q \to P^k = \text{complex projective } k\text{-space}$$

be defined by $\pi(z_1, \cdots, z_n) = [z_1, \cdots, z_{k+1}]$. Let

$$Y_m = \{z \in Y: |z_1|^2 + \cdots + |z_{k+1}|^2 \geq 1/m\} \qquad \text{for } m = 1, 2, \cdots.$$

Since $\pi \mid Y_m$ satisfies a Lipschitz condition of order 1 and P^k is a compact 2k-dimensional Riemannian manifold, we conclude by Lemma 1 that

$$H^\alpha(Y_m \cap \pi^{-1}(w)) = 0 \qquad \text{for } (H^{2k})\text{-almost all } w \in P^k$$

Since $Y = (Y \cap Q) \cup \left(\bigcup_m Y_m\right)$, there exists $w_0 \in P^k$ such that

$$H^\alpha(Y \cap \pi^{-1}(w_0)) = 0,$$

and $P = \pi^{-1}(w_0)$ is our desired (n - k)-plane. ■

In fact, almost all (n - k)-planes P through 0 have the property that $H^\alpha(Y \cap P) = 0$ (see Section 5).

3. REMOVING SINGULARITIES OF FUNCTIONS

In this section we derive measure-theoretic criteria guaranteeing the removability of singularities of analytic functions. In Lemma 3 we state several results that generalize some elementary results in one complex variable. We shall use a special case of Lemma 3 (stated as Lemma 3') in Section 4. Many of the ideas of this section were suggested to the author by Professor Bernard Kripke. Here and in the following sections, we write $0^m = (0, \cdots, 0) \in C^m$.

Definition. For $\alpha > 0$, we say that a mapping $f: X \to Y$ of metric spaces satisfies a *Lipschitz condition of order* α ($f \in L_\alpha$) if

$$\sup_{a \neq b} \frac{d(f(a), f(b))}{d(a, b)^\alpha} < +\infty.$$

The following lemma generalizes some elementary results from the theory of functions of one complex variable.

LEMMA 3. *Let* U *be open in* C^n, *and let* E *be a closed subset of* U. *Let* f *be an analytic function on* $U - E$. *Then* f *can be extended to an analytic function on* U *if one of the following conditions is satisfied:*

(i) $H^{2n-2}(E) = 0$;

(ii) f *is bounded and* $H^{2n-1}(E) = 0$;

(iii) f *can be extended to a continuous function on* U, *and* $H^{2n-1}(E) < +\infty$;

(iv) *for some* α *in the range* $0 < \alpha \leq 1$, $f \in L_\alpha$ *and* $H^{2n-1+\alpha}(E) = 0$.

The proof of Lemma 3 for $n = 1$ is elementary: case (i) is a tautology; cases (ii) and (iii) are proved in [1]; and case (iv) can easily be established by the methods of [1]. We now prove Lemma 3 for $n > 1$, utilizing the result for $n = 1$.

In all cases, $H^{2n}(E) = 0$; therefore E is nowhere dense. Consider an arbitrary point p of E, and assume without loss of generality that $p = 0^n$. We first consider cases (i) and (ii). By Lemma 2, we can pick a complex line L through 0^n so that $H^1(E \cap L) = 0$. Rotate coordinates so that $L = \{z_1 = \cdots = z_{n-1} = 0\}$. By Corollary 2 of Lemma 1, we can pick an open disk D about 0 in C^1 such that $0^{n-1} \times \overline{D} \subset U$

and $(0^{n-1} \times \partial D) \cap E = \emptyset$. Since E is closed in U, there exists an open neighborhood W of 0^{n-1} in C^{n-1} such that $W \times \overline{D} \subset U$ and $(W \times \partial D) \cap E = \emptyset$. Define

$$g(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z_1, \cdots, z_{n-1}, \zeta)}{\zeta - z_n} d\zeta \qquad \text{for } z \in W \times D.$$

Consider the complex lines $L_w = w \times C^1$ for $w \in W$. For case (i), let $F = \{w \in W: E \cap L_w \neq \emptyset\}$; and for case (ii), let $F = \{w \in W: H^1(E \cap L_w) > 0\}$. By Corollary 4 of Lemma 1, $H^{2n-2}(F) = 0$. Therefore the set of points $w \in W$ such that

$$g \not\equiv f \text{ on } w \times D - E$$

is an open subset of F, and hence it is empty. Therefore g extends $f \mid (W \times D - E)$. Since g is analytic, these cases are proved.

Now consider cases (iii) and (iv). We can assume that f is a continuous function on all of U. Using the original coordinates, we define g as above, where W and D are chosen so that $W \times \overline{D} \subset U$. For case (iv), let $F = \{w \in W: H^{l+\alpha}(E \cap L_w) > 0\}$, and for case (iii), let $F = \{w \in W: H^1(E \cap L_w) = +\infty\}$. Proceeding as before, we conclude that

$$g = f \mid W \times D.$$

Hence $f \mid W \times D$ is analytic with respect to z_n and is similarly analytic with respect to each z_k $(1 \leq k \leq n)$. Therefore f is analytic in $W \times D$. ∎

We recall the definition of a negligible set.

Definition. Let U be open in C^n. A subset E of U is *negligible in* U if E is closed in U and nowhere dense in U and if for each open subset V of U, each bounded analytic function f on V - E can be extended to an analytic function on V.

Note that we require negligible sets to be closed. In the proof of our main theorem, we shall use case (ii) of Lemma 3, which we now restate.

LEMMA 3'. *Let* U *be open in* C^n. *Then each closed subset of* U *with Hausdorff* (2n - 1)-*measure zero is negligible in* U.

Lemma 3' contains as a special case a theorem of Bochner and Martin [3, Theorem 6, p. 174].

4. REMOVING SINGULARITIES OF ANALYTIC SETS

Before we can prove our main theorem, we need a simple criterion for recognizing analytic sets.

LEMMA 4. *Let* S *and* T *be open in* C^k *and* C^ℓ, *respectively. Let* X *be a closed subset of* $S \times T$, *and let* $\pi: X \to S$ *be the projection into the first* k *coordinates. Assume that* π *is proper, and that there exists a (closed) negligible subset* F *of* S *such that the set* $X_0 = X - \pi^{-1}(F)$ *is dense in* X *and is a pure* k-*dimensional analytic set in* $(S - F) \times T$. *Then* X *is a pure* k-*dimensional analytic set in* $S \times T$.

Our proof of this result is taken from Bishop's proof [2, pp. 293-294] of the result of Remmert and Stein mentioned in Section 1 of this paper. Our terminology is that of [5, Chapter III, Section B]. Assume without loss of generality that S is connected. Let $S_0 = S - F$. Then S_0 is connected, and $\pi_0 = \pi \mid X_0 \colon X_0 \to S_0$ is an analytic cover. Let λ denote the number of sheets of π_0, and write

$$\pi_0^{-1}(p) = \{\not{p}_1, \cdots, \not{p}_\lambda\},$$

each point of $\pi_0^{-1}(p)$ appearing in $\{\not{p}_1, \cdots, \not{p}_\lambda\}$ as often as its branching order indicates, for any $p \in S_0$. Let $z^0 = (p^0, q^0) \in S \times T - X$ be arbitrary. To prove that X is analytic, we shall find an analytic function on $S \times T$ that vanishes on X but not on z^0. Pick a sequence $p^n \in S_0$ such that $p^n \to p^0$. By taking a subsequence, if necessary, we may assume that $\not{p}_i^n \to x_i \in X$ for $1 \leq i \leq \lambda$, since π is proper. Choose a linear function g on $C^{k+\ell}$ such that

$$g(z^0) \neq g(x_i) \quad \text{for } 1 \leq i \leq \lambda.$$

Define $f\colon S_0 \times T \to C$ by

$$f(p, q) = \prod_{i=1}^{\lambda} [g(p, q) - g(\not{p}_i)] \quad \text{for } p \in S_0,\ q \in T.$$

We can rewrite f in the form

$$f(p, q) = g(p, q)^\lambda + \sum_{i=0}^{\lambda-1} a_i(p) g(p, q)^i,$$

where $a_i(p) = \sigma_i(g(\not{p}_1), \cdots, g(\not{p}_\lambda))$, the σ_i being the elementary symmetric polynomials. Since $\pi_0\colon X_0 \to S_0$ is an analytic cover, the a_i are analytic on S_0. Since the a_i are also bounded in every compact subset of S and $F = S - S_0$ is negligible, the a_i extend analytically to S, and therefore we can extend f to an analytic function f' on $S \times T$. Since f vanishes on the set X_0, which is dense in X, f' vanishes on X. Finally,

$$f'(z^0) = f'(p^0, q^0) = \lim f(p^n, q^0) = \lim \prod_{i=1}^{\lambda} [g(p^n, q^0) - g(\not{p}_i^n)]$$

$$= \prod_{i=1}^{\lambda} [g(z^0) - g(x_i)] \neq 0.$$

Therefore X is analytic. Since X_0 is dense in X and of pure dimension k, X must also be of pure dimension k. ■

We now use this lemma to prove our main theorem.

Let U, E, A, and A' satisfy the conditions in the theorem, with $H^{2k-1}(E) = 0$. Let $\mathscr{R}(A)$ denote the set of regular points of A. Let $z^0 = 0^n$ be an arbitrary point of U. For any complex (n - k)-dimensional subspace P of C^n, let π_P denote the orthogonal projection (along P) mapping C^n onto the subspace orthogonal to P. In order to apply Lemma 4, we shall need a P such that $P \cap A'$ has topological dimension zero and $\pi_P \mid A$ has rank k on a dense subset of $\mathscr{R}(A)$. We choose a countable dense subset $\{x_i\}$ of $\mathscr{R}(A)$, and we let Q_i be the tangent plane of $\mathscr{R}(A)$ at x_i regarded as a k-plane through 0^n in C^n (by identifying C^n with the tangent plane to C^n at x_i in the usual way). Since each Q_i has real dimension 2k, and since $A' \subset A \cup E$, the set

$$Y = \left(\bigcup_i Q_i\right) \cup A'$$

has Hausdorff (2k + 1)-measure zero. By Lemma 2, there exists a complex (n - k)-plane P through 0^n in C^n such that $H^1(P \cap Y) = 0$, and therefore

a) $P \cap A'$ has topological dimension zero, and

b) $P \cap Q_i = \{0^n\}$ for each i.

Thus $\pi_P | A$ has rank k at each x_i, and P has the desired properties. Therefore, the set

$$\mathscr{R}'(A) = \{x \in \mathscr{R}(A): \operatorname{rank}_x(\pi_P | A) = k\}$$

is an open dense subset of A. Make an orthogonal change of coordinates (leaving 0^n fixed) such that $P = \{z_1 = \cdots = z_k = 0\}$. Since $P \cap A' = (0^k \times C^{n-k}) \cap A'$ has topological dimension zero, there exists a bounded open neighborhood T of 0^{n-k} in C^{n-k} such that $0^k \times \overline{T} \subset U$ and $(0^k \times \partial T) \cap A' = \emptyset$. Choose an open neighborhood S of 0^k in C^k such that $S \times \overline{T} \subset U$ and $(S \times \partial T) \cap A' = \emptyset$. Let

$$X = A' \cap (S \times T).$$

Then the projection (into the first k coordinates)

$$\pi = \pi_P | X: X \to S$$

is proper. Let $F = \pi(X \cap E)$. Since π is proper and decreases distance, F is closed in S and $H^{2k-1}(F) = 0$. Therefore, by Lemma 3', F is negligible in S. Let

$$X_0 = X - \pi^{-1}(F) = A \cap [(S - F) \times T].$$

We can now apply Lemma 4 to conclude that X is analytic of pure dimension k, provided we verify that X_0 is dense in $A \cap (S \times T)$. Supposing the contrary, let V be a nonempty open subset of $A \cap (S \times T)$ such that $\pi(V) \subset F$, and choose an $x \in V \cap \mathscr{R}'(A)$. Since π has rank k at x, $\pi(V)$ contains a neighborhood of $\pi(x)$, contradicting the fact that F is nowhere dense. Thus the hypotheses of Lemma 4 are satisfied. ■

5. REMARKS

In this section, we show how the methods of Section 2 yield a proof of an integral-geometric inequality stated without proof by Bishop [2, p. 290]. (Stolzenberg sketched a rather difficult proof [10, pp. 39-42] of this inequality.) In Lemma 5 we state Bishop's inequality in a general form.

We denote the (complex) Grassmann manifold of complex k-dimensional subspaces of C^n by $M_{n,k}$. The real dimension of $M_{n,k}$ is 2k(n - k). Let μ be the invariant Borel measure on $M_{n,k}$ associated with a given metric on $M_{n,k}$ that is invariant under the action of the unitary group and normalized so that $\mu(M_{n,k}) = 1$.

LEMMA 5. *There exist universal constants* a_n *(for* $n \geq 2$*) such that*

$$\int^*_{M_{n,n-k}} H^\alpha(Y \cap P)\, d\mu(P) \leq a_n H^{2k+\alpha}(Y)$$

for $\alpha \geq 0$, $0 < k < n$, *and each set* $Y \subset \{z \in C^n: \|z\| \geq 1\}$.

Proof. Fix k and n, and let $M = M_{n,n-k}$. Let

$$T_r(z) = \{P \in M: d(z, P) \leq r\} \qquad \text{for } z \in C^n.$$

Fix a unit vector $e_0 \in C^n$, and let $N = \{P \in M: e_0 \in P\}$. Then $N \simeq M_{n-1,n-k-1}$ is a submanifold of M with real codimension 2k. For $r > 0$, let

$$N_r = \{P \in M: d_M(P, N) \leq r\}.$$

Estimating the volume of a tube about a compact submanifold by the technique of [11], we see that $\mu(N_r)/(cr^{2k}) \to \text{Volume}(N)$ as $r \to 0$ (where cr^{2k} is the volume of a ball of radius r in R^{2k}). Since $T_r(e_0) \subset N_{c'r}$ for $r < 1/2$, where c' is an appropriate constant, we can therefore choose a constant a_n such that $\mu[T_r(e_0)] \leq a_n r^{2k}$ for $r < 1/2$. Now consider any nonempty set $A \subset Y$ with $\delta(A) < 1/2$, and let $a \in A$. Then

$$\int^* \delta^\alpha(A \cap P)\, d\mu(P) \leq \delta^\alpha(A)\, \mu(\overline{B}),$$

where $B = \{P \in M: A \cap P \neq \emptyset\}$. Since $\overline{B} \subset T_{\delta(A)}(a)$, we have the inequalities

$$\mu(\overline{B}) \leq \mu[T_{\delta(A)}(a)] = \mu[T_{\delta(A)/\|a\|}(e_0)] \leq a_n \left[\frac{\delta(A)}{\|a\|}\right]^{2k} \leq a_n\, \delta^{2k}(A).$$

Therefore

$$\int^* \delta^\alpha(A \cap P)\, d\mu(P) \leq a_n\, \delta^{2k+\alpha}(A).$$

Our result follows from this inequality exactly as in the proof of Lemma 1.

Bishop uses the inequality of Lemma 5 to obtain a lower bound on the volume of certain analytic sets [2, Corollary 2, p. 299] which he then uses in the proof of the following result [2, Theorem 3, p. 299]:

THEOREM (Bishop). *Let* U *be open in* C^n, *let* E *be an analytic set in* U, *and let* A *be a pure* k*-dimensional analytic set in* U - E. *If* $H^{2k}(A)$ *is finite, then* $\overline{A} \cap U$ *is analytic in* U.

The following stronger form of Lemma 2 is an immediate consequence of Lemma 5:

COROLLARY. *Let* Y *be an arbitrary subset of* C^n *with* $0 \notin Y$, *and let* $\alpha \geq 0$. *If* $H^{2k+\alpha}(Y) = 0$, *then*

$$H^\alpha(Y \cap P) = 0 \qquad \textit{for almost all } P \in M_{n,n-k}.$$

In fact, if Y is also assumed to be of type $\mathscr{F}_\sigma$, then it can be shown that the set of P such that $H^\alpha(Y \cap P) > 0$ is of type $\mathscr{F}_\sigma$ and is therefore also of the first category (in $M_{n,n-k}$), since a closed set with volume zero is nowhere dense.

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