October 16, 2000

Math 421 – Dynamical Systems and Chaos
Some Project Ideas

Here are some ideas to get you started thinking about the project you have
to complete as part of this class. You should decide on a project by Tuesday
October 24, when you should hand in a brief description of what you intend
to study.

1. Existence and uniqueness results for solutions of differential
equations

Often there are various ways to show existence of solutions to a mathe-
matical problem. From a practical point of view the most interesting ways
are the constructive ways – methods that tell you not only that a solution
exists but that provide approximations to the solution with accuracy as close
as needed.

A good place to read about constructive methods to solve differential
equations is Chapter 4 of Hubbard & West Part I. A possible project out-
line based on this chapter is the following:

• Understand by way of examples that not all differential equations have
solutions, that those that do have solutions are not necessarily unique, and
what the implications of this for real world predictions are.
• Understand the concept of a Lipschitz function, what kinds of functions
are or are not Lipschitz, and why it might be an appropriate condition for
using in existence/uniqueness results for differential equations.
• Understand the “Fundamental Inequality” – what it tells you qualitatively
about solutions of differential equations, how you get existence and unique-
ness results from it and how to prove it.
• Possible further work: understand the connection of these constructive
existence results to the error analysis of numerical methods.

Notes: a good project for a mathy person who would have liked to see more
rigour in the parts of the course about differential equations, but doesn’t
want to get too removed from the real world.
2. Numerical methods for approximating solutions to differential equations

The course should have convinced you that we can’t expect to write down explicit solutions to most nonlinear differential equations. We’ve described some qualitative features of these solutions (e.g. fixed points, behaviour for long times). But what if you really want to “see” a solution to a differential equation? The answer is to use a numerical method to approximate the solution to a differential equation.

One place to start reading about numerical methods for solving DEs is Chapter 3 of Hubbard & West Part I. A possible project outline based on this chapter is the following:

- What is a numerical method and why are they useful?
- Some simple numerical schemes. e.g. Euler, mid-point Euler, Runge-Kutta.
- Error analysis for numerical schemes and its importance.
- Basic properties of numerical schemes: order, step-size, explicit, implicit, stability.
- Analyzing the effects of finite accuracy arithmetic when implementing numerical schemes on a digital computer.
- Making practical use of these ideas by implementing various numerical schemes on a computer and investigating their behaviour.

Notes: a good project for anyone who needs to solve DEs numerically and wants to understand how people (and computer programs) go about doing this. Definitely should have some computation as part of the project.

3. Structural Stability

In class we have discussed the notion of stability of fixed and periodic points. Another important notion is whether or not an entire dynamical system is in some sense stable – this concept is given the name structural stability. In somewhat vague terms a dynamical system is structurally stable if every “nearby” dynamical system has essentially the same dynamics.

The notion of structural stability is important in applications to the real world. To describe the physical world in terms of mathematics we write down certain models of a physical situation and write down equations that tell us how these models will behave. But there maybe parameters in the model which need to be experimentally determined and hence will only
be known up to some finite accuracy. In other words there may be some uncertainty about exactly what dynamical system should be describing our physical situation. A structurally stable dynamical system is one in which we can be sure that as long as our experimental errors are not too big then we can rely on the answers from our dynamical system – even if we have not used exactly the “true” values of the parameters. If a system is not structurally stable a small error in estimating the parameters may make a big difference to the dynamics.

It would be nice if all interesting physical systems were structurally stable. Unfortunately, that is not the case and there are systems like the Lorenz equations from meteorology that are not even “close” to being structurally stable. Nevertheless, the concept is important in applications.

The goal for this project is to understand the basic ideas related to structural stability for discrete dynamical systems. The starting point is to make precise the notions of “nearby dynamical system” and “essentially the same dynamics”. Pursuing the first idea leads to the notion of two maps being $C^r$-close and the second to the notion of two dynamical systems being topologically conjugate. A reasonable place to start is Section 1.9 of Devaney’s Intro to Dynamical Systems.

- The $C^r$-distance between two maps
- Topologically conjugate dynamical systems
- $C^r$-structural stability of a discrete dynamical system
- Basic examples of structurally stable and unstable systems
- One-dimensional structurally stable dynamical systems on the circle and Morse-Smale maps (Sections 1.14-1.15)
- Structural stability in higher dimensional systems – the Horseshoe Map (Section 2.3)
- Necessary conditions for structural stability – transversality of stable and unstable manifolds of saddle points (Section 2.7)

4. “Strange” Attractors

In both 1d continuous and discrete dynamical systems we saw that fixed points could be attractive – nearby points get closer and closer to the fixed point. In 1d discrete systems we also had periodic orbits and saw that these too could be attracting. In 2d discrete systems and 3d continuous systems it is possible to have other more exotic “attractors” which are geometrically very complicated shapes.

Roughly speaking an attractor is some set of points which the dynamical
system maps to itself, but to which nearby points get closer and closer. Thus in computer simulations of a dynamical system attractors will be a prominent feature. An attractor which is not a fixed point or a periodic orbit is sometimes called \textit{strange}.

The basic goal of this project is to understand precisely what an attractor is and to study some basic examples of “strange” attractors. Section 2.5 of Devaney’s Intro to Dynamics is a reasonable place to start.

- Trapping regions and invariant sets of dynamical systems
- Definition of an attractor
- Basic examples of “strange” attractors and their properties – the solenoid and the Plykin attractor
- Dynamics on attractors and symbolic dynamics
- Other attractors – the Henon (see section 2.9), Lozi and Lorenz attractors
- Computer generated pictures of various attractors.

5. \textbf{Quantitative measures of chaos}

A hallmark of a “chaotic” dynamical systems is sensitive dependence on initial conditions. This means that two nearby points will typically end up far apart soon afterwards. Since in practice one can only make measurements with finite accuracy, this means that for a chaotic system there is some timescale after which we should stop trusting our results. Obviously, quantifying such an notion has useful practical implications. One way to do this is to introduce the idea of a Lyapunov exponent.

The goal of this project is to get a good theoretical and practical understanding of the Lyapunov exponent of a system, how to calculate it exactly in special cases, how to determine it numerically in other cases and what its physical significance is. A reasonable place to start for the theoretical side is Section 2.1 of Guick’s Encounters with Chaos. For the practical side see Section 3.7 and 6.4 of Moon, Chaotic and Fractal Dynamics and the references therein.

- Definition of the Lyapunov exponent for a 1d discrete dynamical system
- Exact calculation of the Lyapunov exponent for some examples: e.g. the tent map, the doubling map, the logistic map (see also Moon section 3.7)
- The meaning of the Lyapunov exponent, its interpretation as “rate of information loss” and its relation to chaotic behaviour (see Moon section 3.7)
- Ways to measure numerically the Lyapunov exponent of a known dynamical system or from experimental time series data. Problems and trustworthi-
nes of such methods especially for experimental data (see section 6.4 Moon)


Here is a quote from Robert May, a distinguished mathematical biologist:

“Given that simple discrete dynamical systems can produce time series that superficially look like random noise, we must look at all such apparently noisy time series with fresh eyes. This is, as it were, the flip side of the chaos coin.”

What May means is that given some irregular experimental time series (e.g. number of cases of measles, cardiac rhythms, various economic data) we would like to be able to tell whether we are seeing “real randomness” or perhaps a low-dimensional but chaotic system. So far, there is no one good answer to this question. But one approach is the so-called method of “non-linear forecasting”.

The essential idea in this approach is to use a library of past patterns to make short-term predictions. By comparing the actual and predicted time series, we can make tentative distinctions between dynamical chaos and noise or experimental errors – for a chaotic dynamical system the accuracy of the prediction should fall off with increasing prediction-time interval (at a rate which gives an estimate of the Lyapunov exponent), whereas for noise the accuracy should be roughly independent of prediction interval.

This project is somewhat less focused than the previous ones. Start by reading the relevant section of the paper by May titled “Necessity and Chance: Deterministic Chaos and Evolution”. Then use the references in that paper to pick a more detailed paper on the subject to form the backbone of your project.
References