Zeros of random polynomials with prescribed Newton polytope

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Theme: Zeros of random sparse polynomials of large degree tend to be concentrated in certain regions.
We begin with 0-dimensions.

**Bézout’s Theorem:** The number of common zeros in $\mathbb{C}^m$ of $m$ polynomials

$$f_1, f_2, \ldots, f_m \in \mathbb{C}[z_1, \ldots, z_m]$$

is \( \leq D := \deg f_1 \deg f_2 \cdots \deg f_m \). Generic polynomials have exactly $D$ common zeros.
A Newton polytope:

\[ P = \{(0,0), (1,0), (0,1), (1,1), (2,0)\} \]

\[
\begin{align*}
 f(z_1, z_2) &= \sum_{\alpha \in P} c_\alpha z^\alpha \\
 &= \sum_{\alpha \in P} c_{\alpha_1 \alpha_2} z_1^{\alpha_1} z_2^{\alpha_2} \\
 &= c_{00} + c_{10} z_1 + c_{01} z_2 + c_{11} z_1 z_2 + c_{20} z_1^2
\end{align*}
\]
Kouchnirenko’s Theorem:

for generic polynomials $f_1, f_2, \ldots, f_m$ with Newton polytope $P$,

$$\# \{ z \in (\mathbb{C}^*)^m : f_1(z) = \cdots = f_m(z) = 0 \} = m! \cdot \text{Vol}(P) .$$

When $P = p\Sigma$,

$$\text{Vol}(P) = p^m \cdot \text{Vol}(\Sigma) = \frac{1}{m!} p^m ,$$

and we get Bézout’s Theorem.

(Generalization: BKK Theorem with ‘mixed volumes’)
For our example,

\[ \text{Vol}(P) = \frac{3}{2}, \ #\text{zeros} = 2! \text{Vol}(P) = 3. \]

Bézout bound = 4.

Where does the extra zero go?
(Answer: to \( \infty \))

What happens to the distribution of the other 3 zeros? (Before giving the answer, we must define the probability measures.)
Let us first describe the probability measure on the space $\mathcal{P}_N$ of all polynomials of degree $N$, i.e. with Newton polytope $N\Sigma$.

$(\Sigma = \text{the standard unit } m\text{-simplex in } \mathbb{R}^m.)$

We identify $\mathcal{P}_N$ with the space of homogeneous polynomials of degree $N$ in $z_0, \ldots, z_m$, i.e. the space

$$H^0(\mathbb{C}P^m, \mathcal{O}(N))$$

of holomorphic sections of the $N^{th}$ power of the hyperplane section bundle $\mathcal{O}(1) \rightarrow \mathbb{C}P^m$. 

6
A random polynomial of degree $N$:

$$f = \sum_{|\alpha|=N} c_{\alpha} \frac{z^\alpha}{r_\alpha} = \sum_{|\alpha|=N} c_{\alpha} \frac{z_0^{\alpha_0} \cdots z_m^{\alpha_m}}{r_\alpha}$$

where

$$r_\alpha = \|z^\alpha\|_{L^2(S^{2m+1})} \sim \left( \binom{N}{\alpha} \right)^{-\frac{1}{2}},$$

and the $c_\alpha$ are independent complex Gaussian random variables with mean 0 and variance 1.

Thus, the probability measure on $\mathcal{P}_N$ is the complex Gaussian

$$d\gamma = \frac{1}{\pi^{d_N}} e^{-\|c\|^2} dc.$$
$d\gamma$ is an SU($m + 1$)-invariant measure on $\mathbb{CP}^m$; physicists call these polynomials the “SU($m + 1$)-polynomials”.

Since the measure is SU($m + 1$)-invariant, so is the expected distribution of zeros.

We let $Z_{f_1,\ldots,f_m}$ denote the sum of $\delta$ measures at the simultaneous zeros of $f_1, \ldots, f_m \in \mathcal{P}_N$.

Then the expected value $\mathbf{E} Z_{f_1,\ldots,f_m}$ is also SU($m+1$)-invariant; therefore

$$\mathbf{E} Z_{f_1,\ldots,f_m} = N^m \omega^m_{FS}.$$
Remarks:

- (Sh - Zelditch 1999, 2002) The zeros are ‘self-averaging’; i.e.,
  \[
  \text{Var}(N^{-m}Z_{f_1},...,f_m) = O(1/N^2)
  \]
  and therefore
  \[
  N^{-m}Z_{f_1},...,f_m \rightarrow \omega^m_{FS} \quad a.s.
  \]

- (Bleher - Sh - Zelditch 2000, 2001) The zeros are correlated on a
  length scale of \(1/\sqrt{N}\); zeros repel in dimension 1 (Hannay 1996);
  zeros attract in dimension \(\geq 3\).

But when we restrict to \textbf{sparse} polynomials, we no longer have uniform
distribution of zeros.
Let $P \subset p\Sigma$ be a convex integral polytope.

We let $\mathcal{P}_P \subset \mathcal{P}_p$ denote the polynomials with Newton polytope $\subset P$. We let $\gamma_{|P}$ denote the conditional probability measure induced from the $\text{SU}(m+1)$-invariant measure on degree $p$ polynomials.

I.e., a random polynomial with Newton polytope $P$ is a polynomial

$$f = \sum_{\alpha \in P} c_{\alpha} \frac{z^\alpha}{r_{\alpha}}$$

where the $c_{\alpha}$ are independent Gaussian random variables with mean 0 and variance 1.
We are interested in polynomials of high degree; so we look at the asymptotics as we expand the polytope.

**Theorem** (Sh - Zelditch 2002). There is a classically allowed region \( \mathcal{A}_P \subset (\mathbb{C}^*)^m \) such that

\[
\frac{1}{(Np)^m} \mathbb{E}_{\gamma|NP}(Z_{f_1,...,f_m}) \rightarrow \begin{cases} 
\omega_{FS}^m, & z \in \mathcal{A}_P \\
0, & z \notin \mathcal{A}_P
\end{cases}
\]

This is consistent with Kouchnirenko’s theorem.
Now consider the ensemble of $m$ independent random polynomials with Newton polytope $P$, equipped with the conditional probability (5). We let $E_{i|P}(Z_{f_1,\ldots,f_m})$ denote the expected density of their simultaneous zeros. It is the measure on $\mathbb{C}^m$ given by

$$E_{i|P}(Z_{f_1,\ldots,f_m})(U) = \int d\gamma_{i|P}(f_1) \cdots \int d\gamma_{i|P}(f_m) \left[ \# \{ z \in U : f_1(z) = \cdots = f_m(z) = 0 \} \right],$$

for $U \subset \mathbb{C}^m$, where the integrals are over $\text{Poly}(P)$. In fact, $E_{i|P}(Z_{f_1,\ldots,f_m})$ is an absolutely continuous measure given by a $C^\infty$ density (see Prop. 4.1). Our first result shows that, as the polytope $P$ expands, these zeros are expected to concentrate in the classically allowed region and are (asymptotically) uniform there:

**Theorem 1.** Suppose that $P \subset p\Sigma \subset \mathbb{R}^m$ is a convex integral polytope with nonempty interior. Then

$$\frac{1}{(Np)^m} E_{i|NP}(Z_{f_1,\ldots,f_m}) \to \begin{cases} \omega_{FS}^m \quad &\text{on } A_P \\ 0 \quad &\text{on } \mathbb{C}^m \setminus A_P \end{cases},$$

in the measure sense; i.e., for any open $U \subset \mathbb{C}^m$, we have

$$\frac{1}{(Np)^m} E_{i|NP}(\# \{ z \in U : f_1(z) = \cdots = f_m(z) = 0 \}) \to m! \text{Vol}_{CP}(U \cap A_P).$$

In fact, our results imply that the convergence of the zero current on the classically allowed region is exponentially fast in the sense that

$$E_{i|NP}(Z_{f_1,\ldots,f_m}) = (Np)^m \omega_{FS}^m + O \left( e^{-\lambda N} \right) \quad \text{on } A_P,$$

for some positive continuous function $\lambda$ on $A_P$.

The following illustration shows the classically allowed region (shaded) and the classically forbidden region (unshaded) when $P$ is the unit square in $\mathbb{R}^2$ (and $p = 2$):

![Figure 1. The classically allowed region for $P = [0, 1] \times [0, 1]$](image)
Description of $A_P$:

Let $\mu : (\mathbb{C}^*)^m \to \mathbb{R}^m$ be the projective moment map given by

$$\mu(z) = \left( \frac{|z_1|^2}{1 + \|z\|^2}, \ldots, \frac{|z_m|^2}{1 + \|z\|^2} \right).$$

The image of $\mu$ is the interior of the standard unit simplex $\Sigma$ in $\mathbb{R}^m$.

Then $A_P = \mu^{-1}(\frac{1}{p}P)$. 
Therefore
\[ s_1 = \frac{a}{n-a}, \quad \tilde{s}_2 = \frac{1}{n-a} = \frac{s_1}{a}, \quad a = \frac{ns_1}{1+s_1} = \frac{n|z_1|^2}{1+|z_1|^2}. \]

In particular,
\[ a < 1 \Leftrightarrow |z_1|^2 < \frac{1}{n-1} \]
and therefore
\[ \mathcal{R}_F = \left\{ (z_1, z_2) : |z_2|^2 \geq \frac{|z_1|^2 + 1}{n}, \ |z_1|^2 < \frac{1}{n-1} \right\}. \]

We have
\[
\log(1 + \|e^{\tau z/2} \cdot z\|^2) = \log \left(1 + |z_1|^2 + \frac{1 + |z_1|^2}{n|z_2|^2} |z_2|^2\right) = \log(1 + |z_1|^2) + \log \frac{n+1}{n},
\]
\[
\langle q(z), \tau_z \rangle = \left\langle \left(\frac{n|z_1|^2}{1 + |z_1|^2}, 1\right), \left(0, \log \frac{1 + |z_1|^2}{n|z_2|^2}\right)\right\rangle = \log(1 + |z_1|^2) - \log |z_2|^2 - \log n.
\]
Therefore
\[ u_\infty = \log |z_2|^2 + n \log(1 + |z_1|^2) + (n+1) \log(n+1) - n \log n. \]
Hence,
\[
\psi_F = \frac{n \sqrt{-1}}{2\pi} \partial \bar{\partial} \log(1 + |z_1|^2)
\]
\[ e^{-b(z)} = \frac{\frac{1}{n+1} |z_2|^2(1 + |z_1|^2)^{n}}{(1 + |z_1|^2 + |z_2|^2)^{n+1}} \quad \text{for} \quad z \in \mathcal{R}_F. \]
Now suppose that \( z \) is a point in \( \mathcal{R}_{F'}. \) Since \( \tau_z \perp T_{F'}, \) we can write \( \tau_z = (\tau_1, n\tau_1). \) Let
\[ \frac{1}{n+1} q(z) = \mu_{\Sigma}(e^{\tau_1 z/2}, z) = \left(x_1, \frac{1}{n}(1 - x_1)\right) \in F'. \]
As before, we write
\[ s_1 = |z_1|^2, \quad s_2 = |z_2|^2, \quad \tilde{s}_1 = |e^{\tau_1} z_1|^2 = e^{\tau_1} s_1, \quad \tilde{s}_2 = |e^{n\tau_1} z_2|^2 = e^{n\tau_1} s_2. \]
1 (and higher) dimensions.

We let $Z_f$ denote the current of integration over the zero set of $f$:

$$(Z_f, \varphi) = \int_{\{z: f(z) = 0\}} \varphi$$

Let us consider polynomials on $\mathbb{C}^2$. Then the zero set is a complex algebraic curve—real dimension 2—its image under the 'Log map' is an amoeba in $\mathbb{R}^2$.

For the case $P = N\Sigma$, we have

$$\mathbf{E} \left( \frac{1}{N} Z_{f_N} \right) = \omega_{FS}.$$
For any integral polytope $P$, we have

**Theorem.**

$$\frac{1}{N_p} \mathbb{E}_{\gamma|NP}(Z_f) \rightarrow \begin{cases} 
\omega_{FS}, & z \in \mathcal{A}_P \\
\psi_P, & z \notin \mathcal{A}_P 
\end{cases}$$

($\psi_P$ is a piecewise $C^\infty$ positive $(1,1)$-form.)
Methods

- **Szegö kernel** and the Poincaré-Lelong formula:

The Szegö kernel

\[
\Pi_{NP}^0(z, w) = \sum_{\text{ortho.basis}} \varphi_j(z)\overline{\varphi_j(w)}
\]

\[
= \sum_{\alpha \in P} \frac{1}{r_{\alpha}^2} z^\alpha \overline{w}^\alpha.
\]

\[
\Pi_{NP}(z, w) = \frac{\Pi_{NP}^0(z, w)}{(1 + \|z\|^2)^{p/2} (1 + \|w\|^2)^{p/2}}
\]

Poincaré-Lelong formula \[\rightarrow\]

\[
E_{P}(Z_s) = \frac{i}{2\pi} \partial \overline{\partial} \log \Pi_P^0(z, \overline{z})
\]
The mass density of $f$

$$\mathcal{M}(z) = \frac{|f(z)|^2}{(1 + \|z\|^2)^p}$$

The expected mass density is

$$\mathbb{E}|_P \mathcal{M}(z) = \frac{1}{\#P} \Pi|_P(z, z).$$

Asymptotics:

$$\mathbb{E}|_{NP} \mathcal{M}(z) = \frac{p^m}{\text{Vol}(P)} + O\left(\frac{1}{N}\right), \quad z \in \mathcal{A}_P$$

$$\mathbb{E}|_{NP} \mathcal{M}(z) \leq e^{-\lambda N}, \quad z \notin \mathcal{A}_P$$

Thus we call the complement of $\mathcal{A}_P$ the classically forbidden region.
• Euler-MacLaurin formulas
  (Khovanskii-Pukhlikov, Brion-Vergne)

The polytope character:
\[ \chi_{NP}(e^w) = \sum_{\alpha \in NP} e^{\langle w, \alpha \rangle}, \quad w \in \mathbb{C}^m. \]

If \( P \) is a simple polytope and \( \|w\| < \epsilon \),
then (Brion-Vergne):
\[ \chi_{NP}(e^w) = N^m \text{Todd} \left( \int_{P(h)} e^{N\langle w, x \rangle} dx \right) \bigg|_{h=0} \]

**Proposition** (Sh-Zelditch).
\[ \chi_{NP}(e^w) = \int_P e^{N\langle w, x \rangle} A(w, x, N) \, dx \]

\( A \) is a polynomial of degree \( m \) in \( N \).
The analysis

The polytope character sifts out the monomials in the polytope, giving us the Szegö kernel:

$$
\Pi_{\mid NP}(z, z) = \int_{T^m} \Pi_{NP\Sigma}(z, e^{i\varphi} \cdot z) \chi_{NP}(e^{i\varphi}) \frac{d\varphi}{(2\pi)^m}
$$

We then use the proposition to write $\Pi_{\mid NP}(z, z)$ as a complex oscillatory integral over $T^m \times P$.

If $z$ is in the allowable region, we have a (unique) non-degenerate critical point of the phase in the interior of $P$ and the result follows from the method of stationary phase.

If $z$ is in the forbidden region, then we must deform the contour, and we obtain a critical point on the boundary of $P$. 
A final remark:

The motivating ideas came from toric geometry. To the polytope \( P \), there is associated a Kähler toric variety \( M_P \).

\( M_P \) is also a symplectic manifold, with a moment map

\[
\mu_P : M_P \longrightarrow P
\]

We then have an oscillatory integral of the form:

\[
\Pi|_{NP}(z, \bar{z}) = \int_{T^m} \int_{M_P} e^{N\Psi(\varphi, \mu_P(w); z)} \times A(N, \varphi, w) \, dw \, d\varphi
\]

We originally followed this alternate approach.